M.A/M.Sc. (Sem – III) Examination, 2010
MATHEMATICS
MT – 703 : Mechanics (New Course)
(2008 Pattern) (Optional)

Time: 3 Hours Max. Marks : 80

N.B. *: 1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Define the following terms :

i) Holonomic constraint

ii) Rheonomic constraint

iii) Scleronomic constraint

Give an example of each type of contraint stated above. **6**

b) Derive Lagrange's equations of motion from D'Alembert's principle. **10**

2. a) Define generalised momentum and show that in absence of non-potential forces, the momentum conjugate to a cyclic coordinate is conserved. **4**

b) For 2-dimensional harmonic oscillator the Hamiltonian is of the form :

H(x, y, Px, Py) = $\frac{1}{2m}(P_x^2 + P_y^2) + \frac{1}{2}k(x^2 + y^2)$, where k and m are constants.

Show that $\frac{1}{2m}(P_x^2 - P_y^2) + \frac{1}{2}k(x^2 - y^2)$ is a constant of motion. **6**

c) Given $L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - k\left(\sqrt{x^2 + b_0^2} - b_0\right)^2$, find Lagrange's equation of motion, if m, k, b_0 are constants. **6**

3. a) Show that if the law of central force is an inverse square law of attraction then the path of the particle is a conic. **10**

b) Show that central force motion is always planar. Further show that the areal velocity in this case is constant. **6**

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4. a) Write a note on Brachistocrone problem. **6**

b) Write Lagrangian for projectile motion. Find its Hamiltonian and Hamilton's equations of motion. **5**

c) Show that $Q = \ln\left(\frac{\sin p}{q}\right)$ and P = q cot p is a canonical transformation. **5**

5. a) If $F(q_i, p_i, t)$ and $G(q_i, p_i, t)$ constants of motion, then show that $[F, G]_{(q, p)}$ is also a constant of motion. **6**

b) Show that if Hamiltonian H of a system does not depend on time explicitly then it represents total energy of the system. **5**

c) Show that identity transformation can not be generated by F_1 or F_4 type of function. **5**

6. a) Define Euler angles. **4**

b) Show that orthogonal transformations are length preserving. **4**

c) Prove the Euler theorem which states that a general displacement of a rigid body with one point fixed is a rotation about some axis. **8**

7. a) Show that finite rotations do not commute. **4**

b) Show that Poisson bracket is invariant under the canonical transformation. **8**

c) Show that $\frac{\partial [F,G]}{\partial t} = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$ **4**

8. a) Show that $Q_1 = q_1$, $P_1 = p_1 - 2p_2$, $Q_2 = p_2$, $P_2 = -2q_1 - q_2$ are canonical transformation. **6**

b) Derive the symplectic condition for a transformation to be canonical. **6**

c) State Kepler's laws of planetary motion. **4**

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M.A/M.Sc. (Sem – III) Examination, 2010
MATHEMATICS
MT – 703 : Functional Analysis
(Old) (Optional)

Time: 3 Hours Max. Marks : 80

Instructions : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) State and prove Hölder's inequality and Minkowski's inequality. **8**

b) On C([a, b]) define the norm as follows :

$$\| f \| = \sup_{a \le x \le b} |f(x)|.$$

Show that C([a, b]) is a Banach space. **8**

2. a) If a normed linear space N is finite dimensional, then prove that every linear operator on N is bounded. **6**

b) Let M be a closed linear subspace of a normed linear space N, and let x_0 a vector not in M. If d is the distance from x_0 to M, show that there exists a functional f_0 in N^* such that f_0 (M) = 0, $f(x_0) = 1$, and $\|f_0\| = \frac{1}{d}$. **6**

c) True/False ? Justify your answer.

If N is complete, then N is reflexive. **4**

3. a) State and prove the open mapping theorem. **7**

b) Prove that a non-empty subset X of a normed linear space N is bounded if and only if f(X) is a bounded set of numbers for each f in N^*. **6**

c) A linear operator S from l_2 into itself is defined by $S(x_1, x_2, x_3 ..) = (x_2, x_3, ...)$. Find the norm of S. **3**

4. a) If x and y are any two vectors in a Hilbert space H, then prove that $|(x, y)| \le \|x\| \, \|y\|$ **6**

b) Show that the parallelogram law is not true in $l_1^n (n > 1)$ **4**

c) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non-zero vector z_0 in H such that $z_0 \perp M$. **6**

5. a) Let $\{e_1, e_2 ..., e_n\}$ be a finite orthonormal set in a Hilbert space H. If x is any vector in H, then prove that

$$\sum_{i=1}^{n} |(x, e_i)| \le \|x\| ;$$

further prove that

$$x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j$$ for each j. **6**

b) Show that an orthonormal set in a Hilbert space is linearly independent. **5**

c) Show that a projection on a Hilbert space H satisfies $O \le P \le I$. Under what conditions will P = O and P = I ? **5**

6. a) Prove that on operator T on a Hilbert space H is self-adjoint if and only if (Tx, x) is real for all x. **6**

b) Show that the unitary operators on a Hilbert space H form a group. **6**

c) Prove that the adjoint operation $T \to T^*$ on *B* (H) has the following properties :

i) $(\alpha T)^* = \overline{\alpha} T^*$; **4**

ii) $(T_1 T_2)^* = T_2^* \, T_1^*$.

7. a) With usual notations prove that $\left(l_p^n\right)^* = l_q^*$. **10**

b) Let y be a fixed vector in a Hilbert space H, and consider the function f_y defined on H, by $f_y(x) = (x, y)$.

Prove that f_y is a linear transformation, and $\|f_y\| = \|y\|$. **6**

8. a) Let the dimension n of a Hilbert space H be 2, let $B = \{e_1, e_2\}$ be a basis for H. Find the spectrum of the operator T on H defined by $Te_1 = -e_1 + 2e_2$ and $Te_2 = e_1 + e_2$. **4**

b) Prove that a closed linear subspace of a Hilbert space H is invariant under an operator T if and only if $M^\perp$ is invariant under T^* . **6**

c) Assuming some hypothesis for on operator T on a Hilbert space H, show that T has a spectral resolution

$T = \lambda_1 P_1 + \lambda_2 P_2 + ... + \lambda_n P_n$. State clearly the assumptions. **6**

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M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS
MT – 706 : Numerical Analysis
(Old Course) (2005 Pattern)

Time: 3 Hours Max. Marks: 80

N.B. *: 1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*
3) ***Use*** *of non-programmable scientific calculators is* ***allowed****.*

1. A) Assume that g(x) and $g'(x)$ are continuous on balanced interval (a,b) = (P – δ, P + δ) that contain the unique fixed point P and starting value P_0 is choosen in the interval. Prove that, if $|g'(x)| \leq k <|$, $\forall x \in [a,b]$, then iteration $P_n = g(P_{n-1})$ converges to P and if $|g'(x)| >|$, $\forall x \in [a,b]$ then iteration $P_n = g(P_{n-1})$ does not converges to P. **6**

B) Investigate the nature of iteration in part (A) when $g(x) = -4 + 4x - \frac{x^2}{2}$.

i) Show that P = 2 and P = 4 are the fixed points.

ii) Use P_0 = 1.a and compute P_1, P_2, P_3. **5**

C) Start with the interval [1.0, 1.8] and use the Bisection method to find an interval of width h = 0.05 that contains a solution of the equation : $e^x - 2 - x = 0$. **5**

2. A) If Newton-Raphson iteration produces a sequence $\{P_n\}_{n=0}^{\infty}$ that converges to root P of function f(n) and if P is simple root ; show that convergence is quadratic and $|e_{n+1}| \approx \frac{|f''(p)|}{2|f'(p)|} |en|^2$ for n sufficiently large. **6**

B) Solve the system of equation, by using the Gauss – Elimination method with partial pivoting.

$$\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$ **5**

C) Let f (x) = $x^3 - 3x+2$ **5**

i) Find Newton – Raphson Formula g (P_{k-1})

ii) Start with $p_0 = 1.2$, compute p_1, p_2, p_3.

iii) Is the sequence of iteration converges quadratically or linearly ?

3. A) Use Newton's method for non-linear system.

$u = f_1 (x,y) = x^2 - y - 0.2$ and $o = f_2 (x, y) = y^2 - x - 0.3$ start with $(p_0, q_0) = (1.2, 1.2)$ and compute (p_1, q_1), (p_2, q_2). **6**

B) Compute divided difference table for $f(x) = 3.2^x$

x :	–1	0	1	2	3
f (x) :	1.5	3	6	12	24

5

C) Find Newton's Polynomial $P_1(x)$, $P_{2(x)}$, $P_3(x)$ and $P_4(x)$

$a_0 = 4$ $a_1 = -1$ $a_2 = 0.4$ $a_3 = 0.01$ $a_4 = -0.002$

$x_0 = 1$ $x_1 = 3$ $x_2 = 4$ $x_3 = 4.5$ $x_4 = 2.5$. **5**

4. A) Assume that $f \in C^{N+1}[a, b]$ and $x_0, x_1 .....x_n \in [a, b]$ are N + 1 nodes. If $x \in [a, b]$ then prove that $f (x) = P_N(x) + E_N(x)$ where $P_N(x)$ is a polynomial that can be used to approximate f(x) and $E_N (x)$ is corresponding error in the approximation. **6**

B) Find the triangular factorization A = LU for the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 4 \\ 2 & -1 & 5 & 0 \\ 5 & 2 & 1 & 2 \\ -3 & 0 & 2 & 0 \end{bmatrix}$$ **5**

C) Consider the system :

$5x - y + z = 10$

$2x + 8y - z = 11$

$-x + y + 4z = 3$ with $P_0 = 0$ **5**

Use Gauss - Seidel iteration to find P_1, P_2 and P_3.

Is this iteration convergences to the solution ?

5. A) Assume that $f \in C^5[a, b]$ and that x-2h, x-h, x, x+h, x+2h$\in$ [a, b]. Prove that :

$$f'(x) \approx \frac{-f(x-2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$$ **6**

B) Let $f(x) = x^3$, find approximation $f'(2)$, use formula in part (A) with h = 0.05. **5**

C) The polynomial $p(x) = -0.02\,x^3 + 0.2\,x^2 - 0.4x + 1.28$. Passes through points (1, 1.06), (2, 1.12), (3, 1.34) and (5, 1.78) find

a) P(4) b) $\int_1^4 P(x)dx$ **5**

6. A) Derive the formula : $f''(x_0) \approx \frac{2f_0 - 5f_{-1} + 4f_{-2} - f_{-3}}{h^2}$ **6**

B) Use sequential trapezoidal rule to compute the approximation T(0), T(1), T(2) and T(3) for the integral $\int_3^8 \frac{1}{x}dx$. **5**

C) Let $f(x) = x + \frac{2}{x}$, use cubic Lagrange's interpolation based on nodes $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 3$ to approximate f(1.5) and f(1.3) **5**

7. A) Use Euler's method to solve the I.V.P.

$y' = 3y + 3t$ over [0, 0.2] with y(0) = 1, h = 0.05

compare with $y(t) = \frac{4}{3}e^{3t} - t - \frac{1}{3}$ **8**

B) Use Runge-Kutta method of order N = 4 to solve I.V.P.

$y^1 = 2t\,y^2$ over [0, 0.2] with y(0) = 1. **8**

8. A) Use Power method to find dominant Eigen Vector and Eigen value for the matrix.

$$A = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$$ **8**

B) Use Householders method to reduce the following symmetric Matrix to trigonal Form.

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$ **8**

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M.A./M.Sc. (Sem. – I) Examination, 2010
MATHEMATICS
(2008 Pattern)
MT-502 : Advanced Calculus (New Course)

Time: 3 Hours Max. Marks: 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define directional derivative. Prove that the scalar field undergoes its maximum rate of change in the direction of the gradient vector. **5**

b) Assume that the partial derivatives $D_1f, D_2f, ... D_nf$ exist in some n-ball $B(\vec{a})$ and are bounded at $\vec{a}$. Then prove that f is continuous at $\vec{a}$. **6**

c) Let $f(x, y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$ whenever $x^2y^2 + (x-y)^2 \neq 0$.

Show that $\lim\limits_{x \to 0}\left[\lim\limits_{y \to 0} f(x, y)\right] = \lim\limits_{y \to 0}\left[\lim\limits_{x \to 0} f(x, y)\right]$

but that f (x, y) does not tend to a limit as (x, y) $\to$ (0, 0). **5**

2. a) State only the implicit function theorem. **8**

Let $\vec{f} = (f_1, f_2)$ be a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $f_1(x, y) = e_x \cos y$ and $f_2(x, y) = e^x \sin y$.

Show that the Jacobian of $\vec{f}$ is not zero in $\mathbb{R}^2$. Let $\vec{a} = \left(0, \dfrac{\pi}{3}\right), \vec{b} = \vec{f}(\vec{a})$ and

let $\vec{g}$ be the continuous inverse of $\vec{f}$ in the neighbourhood of $\vec{b}$ such that $\vec{g}(\vec{b}) = \vec{a}$. Find the explicit formula for $\vec{g}$ and compute $\vec{f}'(\vec{a})$ and $\vec{g}'(\vec{b})$ and verify $\vec{g}'(\vec{b}) = \left[f'(\vec{g}(\vec{b}))\right]^{-1}$.

b) Show that the dot product of two continuous vector fields is continuous. **4**

c) If a scalar field f is differentiable at $\vec{a}$, then show that f is continuous at $\vec{a}$. **4**

3. a) Assume that $\vec{f}$ is differentiable at $\vec{a}$ with total derivative $\vec{T}_{\vec{a}}$, then prove that

$\vec{T}_{\vec{a}}(\vec{y}) = (\nabla f_1(\vec{a}) \cdot \vec{y}, \nabla f_2(\vec{a}) \cdot \vec{y}, \ldots, \nabla fn(\vec{a}) \cdot \vec{y})$

Where $\vec{f} = (f_1, f_2, \ldots, f_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$. **6**

b) Define line integral and illustrate it by an example. Also state the basic properties of the line integrals. **6**

c) Calculate the line integral of the vector field

$\vec{f}(x, y) = (x^2 - 2xy)\,\vec{i} + (y^2 - 2xy)\,\vec{j}$ from (–1, 1) to (1, 1) along the parabola $y = x^2$. **4**

4. a) State and prove the second fundamental theorem of calculus for the line integrals. **6**

b) Let $\vec{f} = (f_1, f_2, \ldots, f_n)$ be a continuously differentiable vector field on an open set S in $\mathbb{R}^n$. If $\vec{f}$ is gradient on S, then prove that the partial derivatives of the components of $\vec{f}$ are related by the equations.

$Difj(\vec{x}) = Djfi(\vec{x})$

for i, j = 1, 2 , ..., n and every $\vec{x} \in S$. **5**

c) Let $S = \mathbb{R}^2 \{(0, 0)\}$ and let $\vec{f}$ be the vector field defined on S by the equation

$$\vec{f}(x, y) = \frac{-y}{x^2 + y^2}\,\vec{i} + \frac{x}{x^2 + y^2}\,\vec{j}$$

Show that $D_1 f_2 = D_2 f_1$ everywhere on S but that, $\vec{f}$ is not a gradient on S. **5**

5. a) Define double integral over general regions. Prove that a continuous function f on a rectangle Q is integrable on Q. **8**

b) Let f be defined on a rectangle $Q = [0, 1] \times [0, 1]$ as follows :

$$f(x, y) = \begin{cases} 1 - x - y & \text{if } x + y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Make a sketch of the ordinate set of f over Q and compute the volume of this ordinate set by double integration. **6**

c) State only Green's theorem for plane regions bounded by piecewise smooth Jordan curves. **2**

6. a) State only the general formula for change of variables in double integrals. Explain the notations used. **5**

b) Let $x = \rho\cos\theta\sin\phi$ $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$, $\rho > 0, 0 \le \Theta < 2\pi$ and $0 \le \phi < \pi$. Show that $J(\rho, \theta, \phi) = -\rho^2 \sin\phi$. **5**

c) Make a sketch of the region of the integration, assuming the existence of the integral evaluate $\iiint_S (1 + x + y + z)^{-3}\, dxdydz$, where S is the solid bounded by the coordinate planes and the plane x + y + z = 1. **6**

7. a) Discuss the independence of the surface integral under change of parametric representation. **6**

b) Define simple parametric surface. If $T = [0, 2\pi] \times [0, \pi/2]$ maps under $\vec{r}(u, v) = a\cos u \cos v\,\vec{i} + a\sin u\cos v\,\vec{j} + a\sin v\vec{k}$ to a surface S, find singular points of this surface. Also explain whether S is simple. **6**

c) In usual notations show that $\text{curl}\,\vec{F} = \nabla \times \vec{F}$. **4**

8. a) State and prove Stoke's theorem. **8**

b) Let S be the surface of the unit cube, $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$, and let $\vec{n}$ be the unit outer normal to S. If $\vec{F}(x, y, z) = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, use the divergence theorem to evaluate the surface integral $\iint_S \vec{F} \cdot \vec{n} ds$. **6**

c) Compute the divergence of $\vec{F}(x, y, z) = (x^2 + yz)\,\vec{i} + (y^2 + xz)\,\vec{j} + (z^2 + xy)\,\vec{k}$. **2**

M.A./M.Sc. Semester – I Examination, 2010
MATHEMATICS
MT – 505 : Ordinary Differential Equations
(New Course) (2008 Pattern)

Time: 3 Hours Max. Marks: 80

*N.B. : 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Verify that $y_1 = e^x$ is one solution of differential equation :

$y'' - f(x)y' + [F(x) - 1]y = 0$. Also find the general solution. **4**

b) Are the function $\phi_1(x) = x^3$ and $\phi_2(x) = x^2|x|$ defined on the interval [–1, 1] linearly independent ? Justify. **4**

c) Discuss the method of undetermined coefficients to find particular solution of the non-homogeneous differential equation $y'' + py' + qy = R(x)$, where p and q are constants. **8**

2. a) State and prove Sturm separation theorem. **8**

b) Find the particular solution of the differential equation $y'' + k^2 y = f(x)$, where K is a positive constant by using method of variation of parameters. **8**

3. a) Find normal form of equation $y'' + P(x)y' + Q(x)y = 0$. **4**

b) Find the particular solution of the initial value problem $y'' + 4y' + 5y = 0$, with y(0)=1 and $y'(0) = 0$. **4**

c) Let u(x) be any non-trivial solution of $u'' + q(x)u = 0$ where q(x) > 0 for all x > 0. If $\int_1^\infty q(x)dx = \infty$, then prove that u(x) has infinitely many zeros on the positive x-axis. **8**

4. a) Find the indicial equation and it's roots for the following differential equations :

   i) $x^3y'' + (\cos^2 x - 1)\,y' + 2xy = 0$

   ii) $2x^2y'' + x(2x+1)y' - y = 0$ **4**

   b) Find the regular singular point of the differential equation

   $$(1-x^2)y'' - 2xy' + p(p+1)y = 0$$ **4**

   c) Find the general solution of $y'' = xy$ in terms of power series in x. **8**

5. a) Use method of Frobenius series to solve the differential equation

   $2xy'' + y' - y = 0$ about regular singular point 0 (zero). **8**

   b) Show that x = ∞ is a regular singular point of $x^2y'' + 4xy' + 2y = 0$. **4**

   c) Replace the following differential equation by an equivalent system of first order equation $y'' + 3y'' + 2y' + y = \sin x$, where y = y(x). **4**

6. a) For the following system.

   i) Find the critical points.

   ii) Find the differential equation of the paths.

   iii) Solve this equation to find the paths and

   iv) Sketch a few of the paths.

   $$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -x + 2y \end{cases}$$ **6**

   b) Find the Liapunov function E(x, y) so that the critical point (0, 0) is a stable critical point of the system :

   $$\begin{cases} \dfrac{dx}{dt} = -2xy \\ \dfrac{dy}{dt} = x^2 - y^3 \end{cases}$$ **5**

   c) Prove that the function $E(x, y) = ax^2+bxy+cy^2$ is of positive definite if and only if a>o and $b^2 - 4ac < 0$. **5**

7. a) Discuss the method of solving the homogeneous linear system with constant coefficients :

$$\begin{cases} \ddot{x}(t) = a_1 x + b_1 y \\ \ddot{y}(t) = a_2 x + b_2 y \end{cases}$$

When the auxiliary equation $m^2 - (a_1+b_2)\, m + (a_1 b_2 - a_2 b_1) = 0$, has real and distinct roots. **8**

b) Solve the following initial value problem by Picards method and compare the result with exact solution :

$$\frac{dy}{dx} = x - y \,;\; y(0) = 1.$$ **8**

8. a) i) State Picard's existence and uniqueness theorem.

ii) Show that the solution of the initial value problem

$$\frac{dy}{dx} = f(x, y) \,;\; y(x_0) = y_0$$

may not be unique although f(x, y) is continuous. **8**

b) Solve the following initial value problem by Picards method :

$$\begin{cases} \dfrac{dy}{dx} = z, \; y(0) = 1 \\ \dfrac{dz}{dx} = -y, \;\; z(0) = 0 \end{cases}$$ **5**

c) Find the general solution of the system :

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = y \end{cases}$$ **3**

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M.A./M.Sc. (Sem. – II) Examination, 2010
MATHEMATICS
MT – 601 : General Topology (New Course)
(2008 Pattern)

Time: 3 Hours   Max. Marks: 80

N.B. : *1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let X be a non-empty set. Show that the collection $\mathcal{T}_F$ = {U⊂ X / X – U is either finite or} all of X. **6**

form a topology on X.

b) Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for the topology $\mathcal{T}$ and $\mathcal{T}'$ respectively on X. Show that $\mathcal{T}$ is finer than $\mathcal{T}$ if and only if for each x∈ X and B∈ $\mathbb{B}$ containing x there exists a basis element $B' \in \mathcal{B}'$ such that $x \in \mathcal{B}' \subset \mathcal{B}$ **5**

c) Show that the topology $\mathbb{R}_l$ is strictly finer than the standard topology on $\mathbb{R}$. **5**

2. a) Show that the collection $S = \left\{\pi_1^{-1}(U)/U \text{ open in } X\right\} U \left\{\pi_2^{-1}(V)/V \text{ open in } Y\right\}$

Forms a subbasis for the product topology on X × Y. **5**

b) Let A⊂X and B⊂Y where X and Y are two topological spaces. Show that $\overline{A \times B} = \overline{A} \times \overline{B}$ in product space X x Y. **6**

c) Show that if A is a subspace of X and B is a subspace of Y then the product topology on A $\times$ B is the same as the topology A $\times$ B inherits as a subspace of X $\times$ Y. **5**

3. a) State and prove the pasting lamma. Show that, if h : $\mathbb{R} \to \mathbb{R}$ be a function defined as h (x) = x if $x \le 0$ and h (x) = $\frac{x}{2}$ if $x \ge 0$ then h is continuous. **6**

b) Show that a subspace of a Hausdorff space is Hausdorff. **5**

c) Let P : X $\to$Y be a continuous map. Show that if there is a continuous map f : Y $\to$X such that Pof equals the identity map of Y then P is a quotient map. **5**

4. a) Show that if the function $f : X \to Y$ is continuous at x, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to f(x). Is the converse true ? Justify. **6**

b) Define 1 the term quotient topology. Give example of quotient maps such that their product is not a quotient map. **5**

c) Let (X, d) be a metric space. Define

$\overline{d} : X \times X \to \mathbb{R}$ as $\overline{d}$ (x, y) = min {d (x, y), 1}

then show that $\overline{d}$ is a metric that induces same topology as d. **5**

5. a) Show that Cartesian product of finitely many connected spaces is connected. **6**

b) Show that a space X is Locally Path connected if and only if for every open set U in X, each path component of U is open in X. **5**

c) Show that no two of the spaces (0, 1), (0, 1], and [0, 1] are homeomorphic. **5**

6. a) Show that every compact subspace of a Hausdorff space is closed. **6**

b) Show that in the finite complement topology on $\mathbb{R}$ every subspace is compact. **5**

c) Justify whether true or false : A topological space X is compact if and only if X is limit point compact. **5**

7. a) Give an example of a Hausdorff space which is not regular. **5**

b) Every compact Hausdorff space is normal. **6**

c) Let X be locally compact and Hausdorff space, $A \subseteq X$, show that A is locally compact whether A is open or closed in X. **5**

8. a) Show that arbitrary product of completely regular spaces is completely regular . **6**

b) State and prove Tychonoff theorem. **10**

B/II/10/380

[3821] – 202

M.A./M.Sc. (Semester – II) Examination, 2010
MATHEMATICS
MT-602 : Differential Geometry (New Course) (2008 Pattern)

Time : 3 Hours   Max. Marks : 80

N.B.: *1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Define a tangent vector at point p to the level set $f^{-1}(c)$ where $f : U \to \mathbb{R}$ is a smooth function and $U \subset \mathbb{R}^{n+1}$. If U is an open set in $\mathbb{R}^{n+1}$ and $p \in U$ is a regular point of f with f(p) = c, then prove that the set of all vectors tangent to $f^{-1}(c)$ at p is equal to $[\nabla f(p)]^{\perp}$. **8**

   b) Show that the graph of any function $f : \mathbb{R}^n \to \mathbb{R}$ is a level set for some function $F : \mathbb{R}^{n+1} \to \mathbb{R}$. **4**

   c) Find the integral curve through the point (1, 0) of the vector field $X(x_1, x_2) = (-x_2, x_1)$. **4**

2. a) Let $S \subset \mathbb{R}^{n+1}$ be a connected n-surface in $\mathbb{R}^{n+1}$. Show that on S there exists exactly two smooth unit normal vector fields N_1 and N_2 and $N_2(p) = -N_1(p)$ for all p in S. **6**

   b) Sketch the level sets $f^{-1}(c)$ for n = 0, 1 of each function given below at the heights indicated.

   i) $f(x_1, x_2, ..., x_{n+1}) = x_1^2 + \frac{x_2^2}{4} + ... + \frac{x_{n+1}^2}{(n+1)^2}$ c = 1.

   ii) $f(x_1, x_2, ..., x_{n+1}) = x_1^2 - x_2^2 - ... - x_{n+1}^2$ c = 2. **5**

   c) Let a, b, c be real numbers such that $ac - b^2 > 0$. Show that the maximum and minimum values of the function $g(x_1, x_2) = x_1^2 + x_2^2$ on the ellipse $ax_1^2 + 2bx_1x_2 + cx_2^2 = 1$ are of the form $\frac{1}{\lambda_1}$ and $\frac{1}{\lambda_2}$ where λ_1, λ_2 are the eigen values of the matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. **5**

3. a) Let $\mathbb{X}$ be a smooth vector field on an open set $U \subset \mathbb{R}^{n+1}$ and $p \in U$. Prove that there exists an open interval I containing 0 and an integral curve $\alpha : I \to U$ of $\mathbb{X}$ such that

i) $\alpha(0) = p$

ii) If $\beta : \tilde{I} \to U$ is any other integral curve of $\mathbb{X}$ with $\beta(0) = p$ then $\tilde{I} \subset I$ and $\beta(t) = \alpha(t)$ for all $t \in \tilde{I}$. **6**

b) Let S be an (n – 1) surface in $\mathbb{R}^n$ given by $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$, U is open in $\mathbb{R}^n$. Show that the cylinder over S is an n-surface. **5**

c) Show by example that the set of vectors tangent at a point p of a level set need not in general be a vector subspace of $\mathbb{R}^{n+1}_p$. **5**

4. a) If S is an n-surface in $\mathbb{R}^{n+1}$, $S = f^{-1}(c)$ where $f : U \to \mathbb{R}$ is such that $\nabla f(q) \neq 0$ for all q in S and if $g : U \to \mathbb{R}$ is a smooth function and p in S is an extreme point of g on S, then prove that there exists a real number λ such that $\nabla(g)(p) = \lambda \nabla f(p)$. **6**

b) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve in S, let $t_0 \in I$ and $V \in S_{\alpha(t_0)}$. Prove that there exists a unique vector field V tangent to S along α which is parallel and has $V_{(t_0)} = V$. **6**

c) Show that if $\alpha : I \to \mathbb{R}^{n+1}$ is a parametrized curve with constant speed then $\ddot{\alpha}(t) \perp \dot{\alpha}(t)$ for all t in I. **4**

5. a) Show that the Weingarten map Lp is self-adjoint. **6**

b) Let S denote the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$, in $\mathbb{R}^3$. Show that α is a geodesic of S if and only if α is of the form $\alpha(t) = (r\cos(at + b), r\sin(at + b), ct + d)$ for some a, b, c, d in $\mathbb{R}$. **6**

c) Let S be an n-surface in $\mathbb{R}^{n+1}$, let $\alpha : I \to S$ be a parametrized curve and X and Y be vector fields tangent to S along α. Show that the covariant derivative satisfies the following property :

$(X + Y)' = X' + Y'$ **4**

6. a) Let S be the n-sphere $x_1^2 + x_2^2 + ... + x_{n+1}^2 = r^2$ of radius r > 0, oriented by the inward unit normal vector field N. Show that the Weingarten map of S is multiplication by $\frac{1}{r}$. **5**

b) Let S be an n-surface in $\mathbb{R}^{n+1}$, oriented by the unit normal vector field N. Let $p \in S$ and $V \in S_p$. Prove that for every parametrized curve $\alpha : I \to S$ with $\dot{\alpha}(t_0) = V$ for some $t_0 \in I$ $\ddot{\alpha}(t_0).N(p) = Lp(V).V$. **6**

c) Let C be the circle $f^{-1}(r^2)$ where $f(x_1, x_2) = (x_1 - a)^2 + (x_2 - b)^2$ oriented by the outward normal. Show that the curvature of C is $\frac{-1}{r}$ at each point. **5**

7. a) For each 1-form w on $U \subset \mathbb{R}^{n+1}$ prove that there exist unique functions $f_i : U \to \mathbb{R}$ i = 1, ..., n + 1 such that $w = \sum_{i=1}^{n+1} f_i dx_i$.
Show that w is smooth if and only if each f_i is smooth. **6**

b) Let $\alpha(t) = (x(t), y(t))$ be a local parametrization of the oriented plane curve C. Show that $K \circ \alpha = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$. **4**

c) Let S be the ellipsoid $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} = 1$

a, b, c all non-zero, oriented by its outward normal. Show that the Gaussian curvature of the ellipsoid is $K(p) = \frac{1}{a^2b^2c^2\left(\frac{x_1^2}{a^4} + \frac{x_2^2}{b^4} + \frac{x_3^2}{c^4}\right)^2}$. **6**

8. a) State and prove inverse function theorem for n-surfaces. **8**

b) Show that the 1-form η on $\mathbb{R}^2 - \{0\}$ defined by $\eta = \frac{-x_2}{x_1^2 + x_2^2} dx_1 + \frac{x_1}{x_1^2 + x_2^2} dx_2$ is not exact. **6**

c) Define a parametrized n-surface in $\mathbb{R}^{n+1}$. **2**

M.A./M.Sc. Sem. – II Examination, 2010
MATHEMATICS
MT – 601 : Real Analysis – II
(Old)
(2005 Pattern)

Time: 3 Hours Max. Marks: 80

N.B. *: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define the total variation of f over [a, b]. If f : [a, b] $\to$ IR satisfies $|f(x) - f(y)| \le K|x - y|$ for all x, y $\in$ [a, b], then prove that f is of bounded variation and $V_a^b f \le K(b - a)$. **6**

b) Let α :[a, b] $\to$ IR be increasing. Prove that a bounded function f: [a, b] $\to$ IR is in $R_\alpha[a, b]$ if and only if, given $\varepsilon > o$, there exists a partition P of [a, b] such that U (f, P), – L (f, P) $< \varepsilon$. **5**

c) If $f \in R_\alpha[a, b]$ then show that $|f| \in R[a, b]$ and $\left|\int_a^b f d^\alpha\right| \le \int_a^b |f| d^\alpha$. **5**

2. a) Let (f_n) be a sequence in $R_\alpha[a, b]$. If (f_n) converges uniformly to f on [a, b], then prove that $f \in R_\alpha[a, b]$. **6**

b) If $f \in R_\alpha[a, b]$, then prove that $\alpha \in R_f[a, b]$. **8**

c) Define α : [0, 1] $\to$ IR by $\alpha(x) = x^2$ and f : [0, 1] $\to$ IR by f(x) = 0 $(0 \le x < \frac{1}{2})$ and f(x) = 1 $(\frac{1}{2} \le x \le 1)$. Find $\int_0^1 f d^\alpha$. **2**

3. a) Show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. **8**

b) Show that $m^*(E) = o$ for any countable set E. **4**

c) Prove that $m^*\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} m^*(E_n)$ for any sequence (E_n) of subsets of IR. **4**

4. a) If E_1 and E_2 are measurable sets, then prove that $E_1 \cup E_2$, $E_1 \cap E_2$ and E_1/E_2 are measurable. **6**

b) Let $f : X \to Y$ be any function. If $\mathcal{B}$ is a 6-algebra of subsets of Y, then show that $\mathcal{A} = \{f^{-1}(B) : B \in \mathcal{B}\}$ is a 6-algebra of subsets of X. **4**

c) Let $f : D \to IR$, where D is measurable set. Then prove that f is measurable if and only if any one of the following holds :

i) $\{f \geq \alpha\}$ is measurable for all real α ;

ii) $\{f < \alpha\}$ is measurable for all real α ;

iii) $\{f \leq \alpha\}$ is measurable for all real α ; **6**

5. a) Let $C \in IR$, and let $f,g: D \to \mathbb{R}$ be measurable functions. Prove that cf and f+g are measurable functions. **6**

b) If $f, g : D \to \mathbb{R}$ are measurable functions, then show that $\{f>g\}$ is measurable. **4**

c) If (f_n) is a sequence of non-negative measurable functions, then prove that

$$\int \left(\lim_{n\to\infty} \inf f_n \right) \leq \lim_{n\to\infty} \inf \int f_n .$$ **6**

6. a) **True/False** ? Justify your answer.

If f is Lebesgue integrable, then f^2 is Lebesgue integrable. **4**

b) Let $f, g \in L_1$. Prove that the following are equivalent :

i) $\int |f - g| = 0;$

ii) $\int_E f = \int_E g$ for every measurable set E. **6**

c) Show that $\lim_{n\to\infty} \int_0^1 f_n = 0$ where $f_n(x) = \dfrac{nx}{1+n^2x^2}$. **6**

7. a) Let KP < ∞. If f, g$\in$ Lp, then prove that f+g$\in$ Lp and

$$\|f + g\|_p^p \leq 2^p\left(\|f\|_p^p + \|g\|_p^p\right).$$ **6**

b) If f : [a, b]$\to$ IR is increasing, then prove that the set of points at which at least one derived number for f is infinite has measure zero. **6**

c) If f$\in$ $L_2[-\pi, \pi]$, then prove that $\|S_n(f) - f\|_2 \to 0$. **4**

8. a) Let l < P < ∞, let f$\in$ L_p(IR), and let ε >o. Then prove that there is an integrable simple function φ with $\|f-\varphi\|_p < \varepsilon$. **6**

b) Prove that f is Lebesgue integrable if and only if |f| is Lebesgue integrable.

Is this result true for Riemann integrable function ? **6**

c) Find Lebesgue integral of $f(x) = \frac{1}{x}$ over interval [0, 1], if exist. **4**

[3821] – 301

M.A./M.Sc. (Sem. – III) Examination, 2010
MATHEMATICS
MT – 701 : Functional Analysis (New)
(2008 Pattern)

Time: 3 Hours Max. Marks: 80

***Instructions** : i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) Let M be a closed linear subspace of a normed linear space N. If the norm of a coset x+M in the quotient space N/M is defined by

$$\| x+M \| = \inf\{\|x+m\| : m \in M\},$$

then prove that N/M is a normed linear space, Further, if N is a Banach space, then prove that N/M is also a Banach space. **8**

b) If M is a closed linear subspace of a normed linear space N, and if T is the natural mapping of N onto N/M defined by T(x) = x+M, show that T is a continuous linear transformation for which $\| T \| \leq 1$. **6**

c) A linear operator S from l_2 into itself is defined by
$S(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...)$.
Find the norm of S. **2**

2. a) State and prove Hanh-Banach theorem. **8**

b) If a Banach space B is reflexive, then prove that B^* is reflexive. **4**

c) True/False ? Justify your answer.
If N is reflexive, then N is complete. **4**

3. a) State and prove Banach-Steinhaus theorem. **8**

b) True/False ? Justify your answer.
If N is Separable, then N^* is separable. **4**

c) Show that the following norms on IR^n are equivalent :

$$\| (x_1, x_2, ..., x_n) \|_1 = \sum_{k=1}^{n} |x_k|$$

$$\| (x_1, x_2, ..., x_n) \|_\infty = \max_{| \leq i \leq n} \{ | x_i | \}$$ **4**

P.T.O.

4. a) Define

$$\| f \|_1 = \int_a^b | f(t) | dt$$

on C ([a, b]). Prove that $\|.\|_1$ is a norm on C ([a, b]).

Does this norm come from an inner product ? Why or why not ? **6**

b) If M is a closed linear subspace of a Hilbert space H, then prove that

$M = M^{\perp\perp}$. **6**

c) Let X = IR^2. Find $M^{\perp}$ if $M = \{(x, y) \mid x + y = 0\} \subseteq IR^2$. **4**

5. a) Let H be a Hilbert space, and let f be on arbitrary functional in H^*. Prove that there exists a unique vector y in H such that f(x) = (x, y) for every x in H. **8**

b) If a Hilbert space H is separable, then prove that every orthonormal set in H is countable. **5**

c) Let H = L_2[a, b] and T : H$\to$H defined by

$$Tu(t) = \int_a^b k(t, s)\, u(s)\, ds.$$

on H define an inner product by

$$(Tu, v) = \int_a^b \int_a^b k(t,s)\, u(s) ds\, \overline{v(t)}\, d(t).$$

Find $T^* u(t)$. **3**

6. a) If T is an operator on a Hilbert space H for which (Tx, x) = 0 for all $x \in H$, then prove that T = 0. **6**

b) If T is an operator on a Hilbert space H, then prove that T is normal if and only if its real and imaginary parts commute. **6**

c) Prove that the adjoint operation $T \to T^*$ on $\mathscr{B}$(H) has the following properties :

i) $T^{**} = T$;

ii) $\| T^* \| = \| T \|$. **4**

7\. a) With usual notations prove that $l_1{}^* = l_\infty$. **7**

b) If P is a projection on a closed linear subspace M of a Hilbert space H, then prove that M reduces on operator T if and only if TP = PT. **5**

c) If T is an arbitrary operator on the Hilbert space H and if α and β are scalars such that $|\alpha| = |\beta|$, then show that $\alpha T + \beta T^*$ is normal. **4**

8\. a) Show that an operator T on a Hilbert space H is normal if and only if its adjoint T^* is a polynomial in T. **6**

b) Let the dimension n of a Hilbert space H be 2, Let B = $\{e_1, e_2\}$ be a basis for H. Find the spectrum of the operator T on H defined by $Te_1 = e_1 - 2e_2$ and $Te_2 = -e_1 - e_2$. **4**

c) Let A and B be normal operators on a Hilbert space H. If A commutes with B^* and B commutes with A^* then prove that A+B and AB are normal. **6**

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[3821] – 301

M.A./M.Sc. (Sem. – III) Examination, 2010
MATHEMATICS
MT – 701 : General Topology (Old)

Time: 3 Hours Max. Marks: 80

N.B. : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1\. a) Let X - be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that for each open set $\cup$ of X and each x in $\cup$, there is a element C of $\mathcal{C}$ such that $x \in C \leq \cup$ then show that $\mathcal{C}$ is a basis for the topology of X. **6**

b) Show that if $\{J_\alpha\}$ is a collection of topologies on X then $\cap J_\alpha$ is also a topology on X . **5**

c) Show that lower limit topology is strictly finer than the usual topology. **5**

2\. a) Define a subspace topology and show that is $\mathcal{B}$ is a basis for a topology on X

then $\mathcal{B}y = \{ B \cap Y \mid B \in \mathcal{B} \}$

is a basis for the subspace topology on Y. **6**

b) Let X = {p, q}, Y = {x, y} be two sets,

$\tau_1 = \{\phi, \{p\}, X\}$, $\tau_2 = \{\phi, \{x\}, \{y\}, y\}$ be the topologies on X and Y respectively.

Compute the product topology on X×Y. **5**

c) Let A, B be subsets of a topological space X and $\{A_\alpha\}$ be a collection in X. Then show that,

i) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

ii) $\overline{\cup A_\alpha} \supseteq \cup \overline{A_\alpha}$. **5**

3. a) Let A be a subset of a topological space X. Prove that

$x \in \overline{A}$ if and only if every open set $\cup$ containing x intersects A. **6**

b) Prove that every map on a discrete space is continuous and every map into an indiscrete space is also continuous. **5**

c) If Y is a subspace of X then show that a separation of Y is a pair of disjoint non-empty sets A and B whose union is Y, neither of which continuous the limit point of other. **5**

4. a) Prove that every second countable space is first countable but converse does not hold. **6**

b) Let f : X $\to$ R and g : X $\to$ R be continuous map.

If $A = \{ x \in X \mid f(x) \le g(x) \}$ and

$B = \{ x \in X \mid f(x) < g(x) \}$ then show that

A is closed and B is open in X. **5**

c) Let X be a topological space $A \subseteq X$,

Let $r : X \to A$ be a continuous map such that r(a) = a for each $a \in A$

show that r is a quotient map. **5**

5. a) Show that the image of a connected space under a continuous map is connected. **5**

b) Prove that every compact subspace of a Hausdroff space is closed. **6**

c) Let X be locally path connected. Show that every connected open set in X is path connected. **5**

6. a) Let Y be a subspace of X, then show that Y is compact if and only if every covering of Y by sets open X contains a finite subcovering of Y. **6**

b) Show that image of a compact space under a continuous map is compact. **5**

c) Let f : X$\to$ Y be continuous. If X has a countable dense subset then show that f(X) has a countable dense subset. **5**

7. a) Let $\{X_\alpha \mid \alpha \in \tau\}$ be a collection of connected spaces with $\bigcap_{\alpha \in J} X_\alpha \neq \phi$.

Prove that $\bigcup_{\alpha \in J} X_\alpha$ is connected. **5**

b) i) State Tube lemmaa. **6**

ii) Give an example of two normal spaces whose product is not normal.

c) Show that a topological space X is compact implies that X is limit point compact but the converse is not true. **5**

8. a) Show that every metrizable topological space is normal. **5**

b) i) State Uryshon Lemma.

ii) Show that a subspace of a regular space is regular. **6**

c) State the following : **5**

i) Tychonoff theorem.

ii) Tietze extension theorem.

[3821] – 305

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS
MT-705 : Graph Theory (New) (2008 Pattern) (Optional)

Time: 3 Hours Max. Marks: 80

N.B. *: 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Prove that an edge of a graph is a cut edge if and only if it belongs to no cycle. **6**

   b) Show that the isomorphism relation is an equivalence relation on the set of simple graphs. **4**

   c) Prove that the complete graph K_n can be expressed as the union of K bipartite graphs if $n \leq 2^K$. **6**

2. a) Let G be a simple graph with vertices $v_1, v_2, ..., v_n$. Let A^k denote the k^{th} power of the adjacency matrix of G under matrix multiplication. Prove that the entry i, j of A^k is the number of v_i, v_j. walks of length k in G. **4**

   b) Prove that every loopless graph G has a bipartite subgraph with atleast $\frac{e(G)}{2}$ edges. **6**

   c) Prove that the center of a tree is a vertex or an edge. **6**

3. a) Prove that a graph G is Eulerian if and only if it has atmost one nontrivial component and its vertices all have even degree. **8**

   b) Use Havel-Hakimi theorem to determine whether the sequence (5, 5, 4, 3, 2, 2, 2, 1) is graphic. Provide a construction or proof of impossibility. **4**

   c) Prove that the non-negative integers $d_1, d_2, ..., d_n$ are the vertex degrees of some graph if and only if $\sum_{di}$ is even. **4**

4. a) Prove that every digraph having no odd cycle has a kernel. **8**

   b) Prove that if T, T' are spanning trees of a connected graph G and $e \in E(T) - E(T')$ then there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree of G. **3**

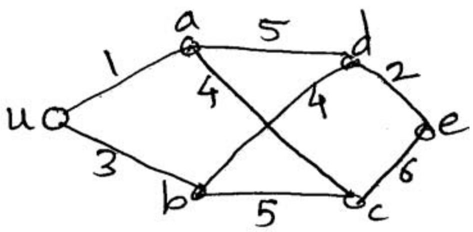
   c) Show that the k-dimensional cube Q_k is a k-regular bipartite graph and find the number of vertices and edges in Q_k. **5**

P.T.O.

5. a) For a set $S \subseteq \mathbb{N}$ of size n, prove that there are n^{n-2} trees with vertex set S. **8**

b) Explain the Kruskal's algorithm for minimum spanning trees. **3**

c) Using Dijkstra's algorithm find the shortest distance from u to every other vertex of the following graph. **5**



6. a) Show that for k > 0, every k-regular bipartite graph has a perfect matching. **5**

b) Prove that if G is a graph without isolated vertices then $\alpha'(G) + \beta'(G) = n(G)$. **8**

c) Let T be a tree with average degree a. Determine n (T) interms of 'a'. **3**

7. a) Prove that the 'Gale-Shapley Proposal Algorithm' produces a stable matching. **6**

b) Prove that if G is a simple graph then $k(G) \leq k'(G) \leq \delta(G)$. **5**

c) Show that two blocks in a graph share atmost one vertex. **5**

8. a) Let G□H denote the Cartesian product of two graphs G and H. Prove that

$X(G \square H) = \max\{X(G), X(H)\}$ **8**

b) Prove that if D is an orientation of G with longest path length $l(D)$ then $X(G) \leq 1 + l(D)$. **8**

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M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS
MT-705 : Rings and Modules (Old) (Optional)

Time: 3 Hours Max. Marks: 80

N.B. *: 1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Define left and right ideals of a ring. Given an example of a left ideal which is not right ideal and an example of right ideal which is not left. **6**

   b) Let R be a ring with unity 1 and I is a left ideal of R such that $I \neq R$, then prove that there is a maximal ideal M of the same kind as I such that $I \subseteq M$. **10**

2. a) Let R be a commutative ring with 1 and I be an ideal in R, then prove that R/I is a field if and only if I is a maximal ideal of R. **6**

   b) If P is prime number, show that P $\mathbb{Z}$ is prime ideal in $\mathbb{Z}$. **5**

   c) Prove that the product of two ideals of the same kind is again an ideal of the same kind. **5**

3. a) Let $f : R \to S$ be a ring homomorphism of R onto S with I = kerf. Prove that R/I is isomorphic to S. **6**

   b) Show that the quotient field of $\mathbb{Z}$ [i] and that of Q [i] is same as Q [i]. **5**

   c) Prove that any division ring D of characteristic O contains Q as the smallest subfield contained in the centre of D. **5**

4. a) Prove that the ring End_k (V) is a simple ring if and only if V is a finite dimensional vector space over a field K. **10**

   b) Let R = M_2 ($\mathbb{R}$) and $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Show that the ideal generated by x is R. **6**

5. a) In a commutative integral domain R with 1 define :

i) irreducible element and

ii) prime element in R. Explain the difference between these two. **6**

b) Prove that the ring $\mathbb{Z}$ [i] of Gaussian integers is Euclidean domain. **8**

c) Show that every Euclidean domain is principal ideal domain. **2**

6. a) Show that every irreducible element in UFD is a prime. **6**

b) Prove that product of two primitive polynomials is primitive. **6**

c) Show that $2 + i\sqrt{5}$ is irreducible in $\mathbb{Z}\left[i\sqrt{5}\right]$ but not prime. **4**

7. a) Define left module. What is unitary left module ? Give an example of non unitary module. **6**

b) Show that unitary modules over $\mathbb{Z}$ are simply abelian groups. **6**

c) Show that intersection of two submodules is again a submodule. **4**

8. a) Define finitely generated module. Let N be a submodule of an R-module M. Prove that there is 1 – to – 1 correspondence between the set of submodules of M/N with the set of submodules of M containing N. **8**

b) Prove that a finitely generated torsion free module over a PID is free. **6**

c) Give two examples of simple module. **2**

M.A./M.Sc. (Sem. – IV) Examination, 2010
MATHEMATICS
MT – 803 : Differential Manifolds (New Course)
(2008 Pattern)

Time: 3 Hours Max. Marks: 80

N.B. *: 1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let M be a K-manifold in $\mathbb{R}^n$ of class C^r. If ∂M is non-empty, prove that ∂M is a K-1 manifold without boundary in $\mathbb{R}^n$ of class C^r. **8**

b) Let $X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{pmatrix}$. With usual notation, find V (X). **4**

c) Show that circle is a 1 manifold in $\mathbb{R}^2$. **4**

2. a) If V is a vector space of dimension n, then find dimension of the space $L^k(v)$. **6**

b) If f (x, y, z) = $3x_1 y_2 z_2 - x_2 y_3 z_1$ and g (u, v) = $2u_4v_1 - 5u_3v_4$ are tensors on $\mathbb{R}^4$, then find $f \otimes g$. **4**

c) Define the wedge product $f \wedge g$. If f (x, y, z) = $2x_2y_2z_1 + x_1y_5z_4$ and g (x, y) = $x_1y_3 + x_3y_1$, then find $f \wedge g$. **6**

3. a) If ω and η are forms of order k and *l* respectively, then prove that $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$. **8**

b) Let $A = \mathbb{R}^2 - \{0\}$. Show that

$\omega = \frac{(x\,dx + y\,dy)}{x^2 + y^2}$ is a closed 1-form on A. Is ω an exact form ? Justify. **8**

P.T.O.

4. a) Let A be open in $\mathbb{R}^k$ and let $\alpha : A \to \mathbb{R}^n$ be of class C^∞. If ω is an l-form defined in an open set of $\mathbb{R}^n$ containing $\alpha(A)$, then prove that $\alpha^*(d\omega) = d(\alpha^*\omega)$. **10**

b) Let A = $(0, 1)^2$ Let $\alpha : A \to \mathbb{R}^3$ be given by $\alpha(u, v) = (u, v, u^2 + v^2 + 1)$. Evaluate $\int_{y_\alpha} x_2\, dx_2 \wedge dx_3 + x_1 x_3\, dx_1 \wedge dx_3$. **6**

5. a) Define volume of a parametrized manifold and prove that it is invariant under reparametrization. **8**

b) Show that the n-ball B^n (a) is an n-manifold in $\mathbb{R}^n$ and $\partial B^n(a) = S^{n-1}(a)$. **4**

c) Let f (x, y) = $x_1y_2 - x_2y_1 + x_1y_1$. Is f an alternating tensor on $\mathbb{R}^3$? Justify. **4**

6. a) For k > 1, prove that if M is a k-manifold with non-empty boundary, then ∂M is orientable. **8**

b) Using a coordinate patch on S^2 (a), the 2-sphere of radius a in $\mathbb{R}^3$, find the area of S^2 (a). **8**

7. a) Let $T : V \to W$ be a linear transformation and $T^* : L^k(W) \to L^k(V)$ be the dual transformation. Prove that for any $f, g \in L^k(W)$, $T^*(f \otimes g) = T^*(f) \otimes T^* g$. **8**

b) Give an example of a map $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ such that α is smooth, α^{-1} is continuous but $(D\alpha)(a)$ has rank less than 2 for some $a \in \mathbb{R}^2$. **4**

c) Find a basis of the tangent space to the sphere S^2 at (0, 0, 1). **4**

8. a) State generalized Stokes' theorem and derive Green's theorem for compact 2-manifold in $\mathbb{R}^2$. **8**

b) Let $\omega = xy\,dx + 2z\,dy - y\,dz$ be a 1-form in $\mathbb{R}^3$. If $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $\alpha(u, v) = (uv, u^2, 3u + v)$, then find $d\omega$, $\alpha^*\omega$, $\alpha^*(d\omega)$ and $d(\alpha^*\omega)$. **8**

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M.A./M.Sc. (Sem. – IV) Examination, 2010
MATHEMATICS
MT – 803 : Measure and Integration (Old Course)

Time: 3 Hours Max. Marks: 80

***N.B.** : i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*
iii) β denotes σ-algebra of subsets of X, μ denotes measure on the measure space (X, β).

1. a) If $E_i \in \beta$, $\mu E_1 < \infty$ and $E_i \supset E_{i+1}$, then prove that $\mu\left(\bigcap_{1=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n$. **5**

b) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \beta$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Prove that there is a unique measurable extended real valued function f on X such that $f \le \alpha$ on B_α and $f \ge \alpha$ on $X \sim B_\alpha$. **6**

c) Show that the collection of locally measurable sets is a σ-algebra. **5**

2. a) State and prove Fatou's Lemma. **6**

b) Prove that the union of a countable collection of positive set is positive. **5**

c) Let F be a bounded linear functional on $L^p(\mu)$ with $1 < p < \infty$. Prove that there is a unique element $g \in L^q$ such that $F(f) = \int fgd\mu$ and $\|f\| = \|g\|_q$. **5**

3. a) If c is a constant and the functions f and g are measurable, then show that the functions f + c, cf, f + g, f.g are all measurable. **4**

b) The class *B* of μ^*-measurable sets is a σ-algebra. If $\bar{\mu}$ is μ^* restricted to *B*, then show that $\bar{\mu}$ is a complete measure on *B*. **8**

c) Define product measure.
Let $\{(A_i \times B_i)\}$ be a countable disjoint collection of measurable rectangles where union is a measurable rectangle $A \times B$. Then prove that $\lambda(A \times B) = \sum \lambda(A_i \times B_i)$ **4**

4. a) Let E be a subset of X such that $E \cap K$ is a Borel set for each compact set K. Then show that E is a Borel set. **5**

b) Let μ be a measure defined on a σ-algebra $\mathscr{M}$ containing the borel sets. If μ is outer regular for each compact set or if μ is inner regular, then prove that μ is regular for each σ-bounded set in $\mathscr{M}$. **6**

c) Show that the intersection of two σ-compact sets is 6-compact. **5**

5. a) Let $\langle A_i \rangle$ be a disjoint sequence of sets in G. Prove that

$$\mu_*\left(E \cap \bigcup_{1=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu_*\left(E \cap A_i\right).$$ **6**

b) If μ^* is a Caratheodory outer measure with respect to Γ, then show that every function in Γ is μ^* measurable. **10**

6. a) For $1 \le p < \infty$ the spaces $L^p(\mu)$ are banach spaces, and if $f \in L^p(\mu)$, $g \in L^q(\mu)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L'(\mu)$ and $\int |fg|\, d\mu \le \|f\|_p \cdot \|g\|_q$. **6**

b) Let F be a monotone increasing function which is continuous on the eight. Prove that there is a unique Bair measure μ such that for all a and b.
$\mu(a, b] = F(b) - F(a)$ **5**

c) Let μ^* be a topologically regular outer measure on X. Prove that each Borel set is μ^*-measurable. **5**

7. a) Let G be transitive topologically equicontinuous group of homomorphisms on a locally compact Hausdorff space X.

Prove that there is a quasi regular Borel measure $\overline{\mu}$ on X which is invariant under G, finite on compact sets and positive on every open set. **10**

b) Let G be a transitive group of homomorphisms on a topological space X. If G is topologically equicontinuous at some X_0 and Y_0, then prove that it is topologically equicontinuous (at each point). **6**

8. a) Find the Hausdorff dimension of the cantor ternary set. **6**

b) State and prove Radon-Mikodym theorem. **10**

M.A./M.Sc. (Semester – IV) Examination, 2010
MATHEMATICS
MT-805 : Lattice Theory (New Course) (2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B.: *1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Let the algebra L = < L ; $\wedge, \vee$ > be a lattice. Set $a \leq b$ if and only if $a \wedge b = a$. Prove that L^P = < L; $\leq$ > is a poset and as a poset it is a lattice. **7**

b) Let I be an ideal and let D be a dual ideal. If $I \cap D \neq \phi$, then show that $I \cap D$ is a convex sublattice and every convex sublattice can be expressed in this form in one and only one way. **6**

c) Is every bounded lattice a complete lattice ? Justify your answer. **3**

2. a) Define the concept of isomorphism in lattice and prove that every isomorphism is an isotone map but not conversely. **6**

b) Define congruence relation on a lattice L. If θ is a congruence relation of L, then prove that for every $a \in L$, $[a]\theta$ is a convex sublattice. **6**

c) Prove that the direct product of chains is a distributive lattice. **4**

3. a) Let L be a pseudo complemented lattice. Prove that $S(L) = \{a^* \mid a \in L\}$ is a lattice. **8**

b) Prove that a lattice is distributive if and only if it is isomorphic to ring of sets. **8**

4. a) State and prove Birkhoff's criterion for distributivity. **12**

b) Find all congruences of five element non-modular lattice. **4**

5. a) State and prove Stone's Theorem for distributive lattices. **8**

b) Prove that a distributive lattice is complemented if and only if the poset P(L) of prime ideals of L is unordered. **8**

6. a) Let L be a lattice of finite length. If L is semimodular then prove that any two maximal chains of L are of same length. **7**

   b) Let L be a lattice and a, b $\in$ L. If aMb and bM*a hold in L. Prove that $[a \wedge b, b] \cong [a, a \vee b]$. **7**

   c) Give an example of a lattice which is semi-modular but not modular. **2**

7. a) State and prove fixed point theorem for complete lattices. **7**

   b) Prove that a bounded conditionally complete lattice is complete. **5**

   c) Prove that the dual of a modular lattice is modular. **4**

8. a) Let L be a semi-modular lattice. Prove that if p and q are atoms of L $a \in L$ and $a < a \vee q \leq a \vee p$. Then prove that $a \vee p = a \vee q$. **5**

   b) Prove that a lattice is modular if and only if it does not contain a pentagon. **8**

   c) Prove that a distributive lattice is modular but not conversely. **3**

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M.A./M.Sc. (Semester – IV) Examination, 2010
MATHEMATICS
MT-805 : Field Theory (Old)

Time : 3 Hours Max. Marks : 80

N.B.: *1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define a finite extension. Define an algebraic extension. Let E be a finite extension of field F. Then prove that E is an algebraic extension of F. **6**

b) Let E = $\mathbb{Q}(\alpha)$, where α is a root of the equation $\alpha^3 + \alpha^2 + \alpha + 2 = -3$. Express α^{-1}, $(\alpha^2 + \alpha + 1)(\alpha^2 + \alpha)$ and $(\alpha - 1)^{-1}$ in the form $a\alpha^2 + b\alpha + c$ with a, b, c $\in \mathbb{Q}$. **6**

c) Give an example of a monic irreducible polynomial of degree 2 and of degree 3 over the field $\mathbb{Z}/7\mathbb{Z}$. **4**

2. a) Let K be an algebraic extension of k, contained in an algebraic closure k^a of k. Then show that following are equivalent :

i) Every embedding of K in k^a over k induces an automorphism of K.

ii) K is the splitting field of a family of polynomials in k [x].

iii) Every irreducible polynomial of k[X] which has a root in K splits into linear factors in K. **10**

b) Let α and β be two elements which are algebraic over F. Let $f(X) = I_{rr}(\alpha, F, X)$ and $g(X) = I_{rr}(\beta, F, X)$. Suppose that degf and degg are relatively prime. Show that g is irreducible in the polynomial ring $F(\alpha)$ [X]. **6**

3. a) Let α be a real number such that $\alpha^4 = 5$.

i) Show that $\mathbb{Q}(i\alpha^2)$ is normal over $\mathbb{Q}$.

ii) Show that $\mathbb{Q}(\alpha + i\alpha)$ is normal over $\mathbb{Q}(i\alpha^2)$.

iii) Show that $\mathbb{Q}(\alpha + i\alpha)$ is not normal over $\mathbb{Q}$. **6**

b) Let K be a normal extension of k. Let G be its group of automorphisms over k, and let K^G be the fixed field of G. Then prove that K^G is purely inseparable over k. **5**

c) Let $f(x) \in k[X]$ be a polynomial of degree n. Let K be its splitting field. Show that [K : k] divides n!. **5**

4. a) Let p be a prime and $n \geq 1$ be an integer. Prove that there exists a finite field of order p^n uniquely determined as a subfield of an algebraic closure $\mathbb{F}_p^a$. Further, show that it is the splitting field of the polynomial $X^{p^r} - X$ and every element of the field is a root of this polynomial. **10**

b) Show that $\sqrt{2} + i$ is algebraic over $\mathbb{Q}$ of degree 4. **6**

5. a) Let K be a Galois extension of k. Let G be its Galois group. Then prove that $k = K^G$. If F is an intermediate field $k \subset F \subset K$, then show that K is Galois extension of F. **8**

b) Let ζ be a primitive n-th root of unity. Show that $\mathbb{Q}(\zeta)$ is a Galois extension of $\mathbb{Q}$ and determine its Galois group. **8**

6. a) Let ζ be a primitive n-th root of unity. Let $K = \mathbb{Q}(\zeta)$ and p be a prime. Such that $n = p^r$ $(r \geq 1)$. Then show that $N_{K/\mathbb{Q}}(1 - \zeta) = p$. **5**

b) Let G be a monoid and K a field. Let $\chi_1, \dots, \chi_n$ be distinct characters of G in K. Then prove that they are linearly independent over K. **7**

c) Define a cyclic extension. State Hilbert's theorem 90. **4**

7. a) Let E/k be a finite extension of a field k. When do you say that E/k is a solvable extension ? When do you say that E/k is solvable by radicals ? If E is a separable extension of k then show that if E/k is solvable extension then E/k is solvable by radicals. **8**

b) Let $E = k(\alpha)$ be a separable extension. Let $f(x) = I_{rr}(\alpha, k, X)$, and let f(X) be its derivative. Let $\frac{f(X)}{X - \alpha} = \beta_0 + \beta_1 X + \dots + \beta_{n-1} X^{n-1}$ with $\beta_i \in E$.

Then prove that the dual basis of $1, \alpha, \dots, \alpha^{n-1}$ is $\frac{\beta_0}{f'(\alpha)}, \dots, \frac{\beta_{n-1}}{f'(\alpha)}$. **8**

8. a) If α is a complex root of $x^6 + x^3 + 1$, find all homomorphisms $\sigma : \mathbb{Q}(\alpha) \to \mathbb{C}$. **5**

b) If p is a prime number then prove that

$$\Phi_p(X) = X^{P-1} + X^{P-2} + \dots + 1$$

Further for an integer $r \geq 1$,

$$\Phi_{p^r}(X) = \Phi_p(X^{p^{r-1}})$$ **6**

c) Show that every element of a finite field can be written as a sum of two squares in that field. **5**

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M.A./M.Sc. (Semester – IV) Examination, 2010
MATHEMATICS
MT-806 : Lattice Theory (Old)
(2005 Pattern)

Time : 3 Hours Max. Marks : 80

N.B.: *i) Answer* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let A be the set of all subgroups of a group G; for X, Y $\in$ A, the set X $\leq$ Y to mean X $\leq$ Y. Prove that < A; $\leq$ >is a lattice. **5**

   b) Define the concept of homomorphism in lattices and prove that every homomorphism is an isotone map but not conversely. **5**

   c) Define congruence relation on a lattice L. If θ is a congruence relation of L, then prove that for every a $\in$ L, [a]θ is a convex sublattice. **6**

2. a) Let L be a lattice and con (L) be the set of all its congruences. Then prove that Con(L) is a lattice. **8**

   b) Prove that L is distributive if and only if $(x \wedge y) \vee (y \wedge z) \vee (z \wedge x) = (x \vee y) \wedge (y \vee z) \wedge (z \vee x)$ holds in L. **6**

   c) Show that $N_S \cong L \times K$ implies that L or K has only one element, where N_S is a non-modular lattice with S elements. **2**

3. a) State and prove stone theorem for distributive lattices. **8**

   b) State and prove Nachbin theorem. **8**

4. a) Let L be a pseudocomplemental lattice. S(L) = {a* | a$\in$ L}. Assuming that S(L) is a lattice, prove that S(L) is a Boolean lattice. **8**

   b) Prove that a lattice is modular if and only if it does not contain a pentagon. **8**

5. a) Prove that every ideal of a lattice L is prime if and only if L is a chain. **5**

   b) Prove that in a Boolean lattice x $\neq$ 0 is a Join irreducible if and only if x is an atom. **5**

   c) Let L be a pseudocomplemental lattice. Show that $a^{**} \vee b^{**} = (a \vee b)^{**}$ **6**

P.T.O.

6. a) Let L be distributive lattice with 0. Verify that every prime ideal p contains a minimal prime ideal θ (that is $p \geq \theta$ and for any prime ideal X of L, $\theta \geq X$ implies $\theta = X$). Is the distributive lattice with 0, a necessary condition to prove the result ? **8**

   b) Prove that a lattice is Boolean if and only if it is isomorphine to a field of sets. **8**

7. a) Prove that every standard element is distributive but not conversely. **5**

   b) Let a be distributive element of a lattice L if and only if (a] is a distributive element of Id L, the ideal lattice of L. **6**

   c) Prove that an element a is standard if and only if $x \leq a \vee \forall$ implies $x = (x \wedge a) \vee (x \wedge y)$. **5**

8. a) Let L be a lattice and D denote the set of all distributive elements of L. Then prove that a, b $\in$ D imply that $a \vee b \in D$. **5**

   b) Prove that set of all standard elements forms a sublattice. **5**

   c) Prove that the following inequalities hold in any lattice L.

      i) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$

      ii) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$

      iii) $(x \wedge y) \wedge (x \wedge z) \leq x \wedge (y \vee (x \wedge z))$. **6**

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M.A./M.Sc. (Sem. – I) Examination, 2010
MATHEMATICS
MT – 501 : Real Analysis – I (New Course)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : *1) Attempt **any five** questions.*
*2) Figures to the **right** indicate full marks.*

1. a) Give examples of two norms on a linear space which are not equivalent. **6**

b) If l^2 denotes the set of all square summable sequences of complex numbers,

then show that $<\{x_n\}, \{y_n\}> = \sum_{n=1}^{\infty} x_n \overline{Y}_n$ is an inner product on l^2. **6**

c) Justify whether the following statement is true or false.
int $(A \cup B)$ = int (A) $\cup$ int (B). **4**

2. a) State and prove Cauchy-Schwarz inequality. **6**

b) Give an example of a closed and bounded set in a metric space that is not compact. Justify your answer. **5**

c) If f and g are measurable functions, then prove that f+g is measurable. **5**

3. a) State and prove Fatou's lemma. **8**

b) Find the Fourier series for the function f(x) = x. **6**

c) State Stone-Weierstrass theorem. **2**

4. a) State and prove Lebesgue's dominated convergence theorem. **8**

b) Assume that μ is a countably additive function defined on a ring R. Let

$A, An \in R\ A_1 \subseteq A_2 \subseteq ....$ and $A = \bigcup_{n=1}^{\infty} An$.

Prove that $\lim_{n\to\infty} \mu(An) = \mu(A)$. **6**

c) State Hölder's inequality. **2**

5. a) State and prove Ascoli – Arzela theorem. **8**

b) Show that the trignometric system

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(mx)}{\sqrt{\pi}} \quad n, m = 1, 2, ...$$

is an orthonormal sequence in $L^2\left([-\pi, \pi], m\right)$. **6**

c) State Parseval's theorem. **2**

6. a) If $\{\mho n\}$ is a sequence of open, dense subsets of a complete metric space M, then prove that $\bigcap_{n=1}^{\infty} \mho n$ is dense in M. **8**

b) Give an example of a set of measure zero which is not of first category. **5**

c) If f is a measurable function, then show that $|f|$ is measurable. **3**

7. Justify whether the following statement is **true** or **false**.

a) If a sequence of Riemann integrable functions {fn} converges to a function f, then f is also Riemann integrable. **5**

b) Prove that the outer measure m* is countably sub additive. **5**

c) Let X be a complete metric space and T a contraction from X into X. show that there exists a unique fixed point of T. **6**

8. a) State and prove the Weierstrass approximation theorem. **12**

b) Show that $\mathbb{R}$ with discrete metric is not separable. **4**

M.A./M.Sc. (Semester – I) Examination, 2010
MATHEMATICS
MT – 503 : Linear Algebra (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) Prove that a linearly independent subset of a finite dimensional vector space can be extended to form a basis of the vector space. **6**

b) Give an example of a vector space with 125 elements. **5**

c) Find a two dimensional subspace of $\mathbb{R}^4$ which does not contain the vectors $[1\ 3\ 2\ 5]^t$ and $[2\ 4\ 3\ 1]^t$. **5**

2. a) Prove that if V and V' are finite-dimensional vector spaces over the same field K then $V \simeq V'$ if and only if $\dim v = \dim v'$. **6**

b) If U, V and W are vector spaces over the same field K and $T \in L(V, W)$, $S \in L(U, V)$ then show that $\text{Ker } S \subseteq \text{Ker } TS$ and $\text{im } TS \subseteq \text{im } T$. **5**

c) If S and T are idempotent linear operators on a vector space V then show that :

i) I – T is idempotent. I is identity operator.

ii) S + T is idempotent if ST = TS = 0. **5**

3. a) If W_1 and W_2 are finite dimensional subspaces of a finite dimensional vector space V, then prove that
$\dim W_1 + \dim W_2 = \dim (W_1 + W_2) + \dim (W_1 \cap W_2)$. **6**

b) i) If V is 2n-dimensional vector space and if W_1 and W_2 are (n + 1) dimensional subspaces of V then prove that $\dim (W_1 \cap W_2) \geq 2$. **3**

ii) If W_1 and W_2 are finite dimensional subspaces of a vector space such that $\dim (W_1 + W_2) = 1 + \dim (W_1 \cap W_2)$ then show that $W_1 + W_2$ is equal to one of the subspaces and $W_1 \cap W_2$ is equal to other. **3**

c) If V is a finite dimensional vector space over the field K and if X and Y are the subspaces of V then prove that $(X \cap Y)° \supseteq X° + Y°$. **4**

P.T.O.

4. a) Let V be a finite dimensional vector space over the field K and let T be a linear operator on V. Then prove that there exist a vector of $y \in V$ such that

$m_T^Y(x) = m_T(x)$. **6**

b) If $V = F_3[x]$ is a vector space of all polynomials of degree ≤ 3 and if $T : V \to V$ is a linear transformation given by $T(f) = f'$ (derivative of f) then find the minimal polynomial of T. **5**

c) If $B = \{[1, 0]^t, [0, 1]^t\}$ is a basis of a vector space $\mathbb{R}^2$ over $\mathbb{R}$, then find a basis of $(\mathbb{R}^2)^*$ dual to B. **5**

5. a) If $T : V \to V$ is a linear operator then define a Jordan chain for T of length K associated with eigen value λ. Prove that a Jordan chain consists of linearly independent vectors. **6**

b) Find the Jordan form for the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$ **5**

c) Give all possible rational canonical forms if the characteristic polynomial is $(x + 2)^2 (x - 3)^2$. **5**

6. a) If V is an inner product space over the field F and if $u, v \in V$ then prove that

$||u + v|| \leq ||u|| + ||v||$ and

$\big| ||u|| - ||v|| \big| \leq ||u - v||$ **6**

b) If V is a finite dimensional inner product vector space and $W \subseteq V$ then define orthogonal complement of W. If W is a subspace of V then prove that

$V = W \oplus W^{\perp}$ **5**

c) Use Gram-Schmidt orthonormalization process to obtain an orthonormal basis spanned by the following vectors in the standard inner product space $\mathbb{R}^2$.

$u = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. **5**

7. a) If V and W are two finite dimensional inner product spaces and $T \in L(V, W)$ then prove that there exist a unique linear transformation $T^* : W \to V$ such that for all $v \in V$ and $w \in W$, $(Tv, w) = (v, T^*w)$. **6**

b) State Riesz representation theorem. If $V = \mathbb{R}_3[x]$ with the inner product

$(p(x), q(x)) = \int_{-1}^{1} p(x)q(x)\,dx$ and if D^* is adjoint of the differential operator D.

Then find the action of D^* on basis element $1 \in V$. **5**

c) If $A \in F^{n \times n}$ is a unitary matrix then prove that the columns of A form an orthonormal basis of the standard inner product space F^n. **5**

8. a) If T is a normal operator on a inner product space V over the field F then prove that

i) $\|Tv\| = \|T^*v\|$ for all $v \in V$.

ii) If for $v \in V$, $Tv = \lambda v$, $\lambda \in F$ then $T^*v = \bar{\lambda} v$.

iii) Eigen vectors corresponding to distinct eigen values of T are orthogonal. **6**

b) Show that any 2×2 orthogonal matrix is of the form $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ or

$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ for some θ. **4**

c) Define positive definite operator.

i) Show that if T is positive definite operator then so is T^{-1}.

ii) If T is a positive definite operator then show that $\langle x, y \rangle = (Tx, y)$, $x, y \in V$ is an inner product on V. **6**

[3821] – 203

M.A./M.Sc. (Sem. – II)Examination, 2010
MATHEMATICS
MT – 603 : Groups and Rings (New Course)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. :1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Show that the set of 2×2 matrices over $\mathbb{R}$ with determinant 1 is a group under multiplication. Is it abelian ? **5**

b) Can a group of order 35 have an element of order 20 ? Justify. **3**

c) Prove the following :

i) Every subgroup of a cyclic group is cyclic.

ii) Moreover, if $|\langle a \rangle| = n$, then the order of any subgroup of $\langle a \rangle$ is a divisor of n.

iii) For each positive divisor K of n, the group $\langle a \rangle$ has exactly one subgroup of order K. **8**

2. a) Show that every permutation of a finite set can be written as a cycle or product of disjoint cycles. **5**

b) Define D_4. List all its elements. Is D_4 abelian ? Justify. **6**

c) Suppose that G is an Abelian group with an odd number of elements. Show that the product of all of the elements of G is the identity. **5**

3. a) State and prove Cayley's theorem. **6**

b) With usual notations, show that for every positive integer $\cap$, Aut (Zn) $\cong \cup_n$. **5**

c) Is the set of even integers isomorphic to the set of multiples of 3 ? Justify. **5**

P.T.O.

4. a) Calculate all the left cosets of $A = \{R_0, R_{180}\}$ in D_4, the Dihedral group of order 8. **5**

   b) Define stabilizer and orbit of a point. Give an example of each. **5**

   c) Determine all cyclic subgroups of order 10 in $Z_{100} \oplus Z_{25}$. **6**

5. a) Show that $Z_2 \oplus Z_2 \oplus Z_2$ has seven subgroups of order 2. **4**

   b) Let G be a group and let Z (G) be the center of G. If G/z(G) is cyclic, then prove that G is abelian. **5**

   c) Show that if G is the internal direct product of finite number of subgroups then it is isomorphic to the external direct product of the same subgroups. **7**

6. a) Give an element of order 10 in A_9. **2**

   b) Define kernel and image of a group homomorphism. Prove that the kernel is a normal subgroup of the domain. Is image a normal subgroup of the codomain ? **8**

   c) List all the elements that describe the rotations of a tetrahedron. **6**

7. a) State and prove the fundamental theorem of finite abelian groups. **14**

   b) State Lagranges Theorem. **2**

8. a) Let G be a finite group and let P be a prime. If P^k divides |G|, then prove that G has atleast one subgroup of order P^k. **6**

   b) Determine all groups of order 66. **6**

   c) Define conjugacy class of ‘a’ and state the class equation for finite groups. **4**

[3821] – 204

M.A./M.Sc. (Sem. II) Examination, 2010
MATHEMATICS
MT – 604 : Complex Analysis (New Course)
(2008 Pattern)

Time: 3 Hours | Max. Marks: 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define a Mobius transformation. Prove that Mobius transformation is the composition of translation, dilation and the inversions. **5**

   b) Let f and g be analytic on open set G and Ω respectively and suppose f(G) $\subset \Omega$. Prove that $g \circ f$ is analytic on G and $(g \circ f)'(z) = g'(f(z))\, f'(z)$ for all z in G. **8**

   c) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} a^n z^n$ **3**

2. a) If G is open and connected and $f : G \to \mathbb{C}$ is differentiable with $f'(z) = 0$ for all z in G, then prove that f is constant. **6**

   b) Let G be either the whole plane " or some open disk. If $u : G \to IR$ is a harmonic function, then prove that u has harmonic conjugate. **5**

   c) Let G be a region and define $G^* = \{ z \mid \bar{z} \in G \}$. If $f : G \to$ " is analytical, prove that $f^* : G^* \to \mathbb{C}$ defined by $f^*(z) = \overline{f(\bar{z})}$ is also analytic. **5**

3. a) If γ [a,b] $\to$ " is piecewise smooth then prove that γ is of bounded variation and $V(\gamma) = \int_a^b |\gamma'(t)|\,dt$. **6**

   b) If $\gamma : (a,b) \to \mathbb{C}$ is a rectifiable path and $\Phi : [c,d] \to [a,b]$ is continuous non-decreasing function $\Phi(c) = a$ and $\Phi(d) = b$, then prove that for any function f continuous on $\{\gamma\}$ $\int_\gamma f = \int_{\gamma o \Phi} f$. **5**

   c) Let $\gamma(t) = e^{it}$ for $0 \le t \le 2\pi$. Then find $\int_\gamma Z^n\, dz$ for every integer n. **5**

P.T.O.

4. a) Let $f : G \to$ " be analytic and suppose $\overline{B}(a, \gamma) \subset G$ ($\gamma > 0$). If $\gamma(t) = a + re^{it}$

$0 \le t \le 2\pi$, then prove that $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w - z} dw$ for $|z - a| < r$. **8**

b) If p (z) is a nonconstant polynomial then prove that there is a complex no; a with p (a) = 0. 4

c) Evaluate the integral $\int_\gamma \frac{\sin z}{z^3} dz$ $\gamma(t) = e^{it}$ $0 \le t \le 2\pi$. **4**

5. a) State and prove maximum Modulus theorem. **6**

b) If $\gamma : [0,1] \to$ " is a closed rectifiable curve and a not on $\{\gamma\}$ $\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}$ is an integer. **5**

c) Let G be a region and $f : G \to$ " be analytic and $a \in G$ such that

$|f(a)| \le |f(z)|$ $\forall$ $z \in G$. Show that either f (a) = 0 or f is constant. **5**

6. a) State and prove Morera's theorem. **6**

b) If G is simply connected and $f : G \to$ " is analytic then prove that f has a primitive in G. **5**

c) Let $\gamma(t) = 1 + e^{it}$ for $0 \le t \le 2\pi$. Find $\int_\gamma \left(\frac{z}{z-1}\right)^n dz$ for all positive integers n.

5

7. a) If f has an isolated singularity at a then prove that the point z = a is a removable singularity if and only if $\underset{z \to a}{\text{sim}}$ (z - a) f (z) = 0. **8**

b) Let f be analytic in the region G except for the isolated singularities a_1, a_2 ----- a_n. If γ is a closed rectifiable curve in G which does not pass through any of

the point a_k and if $\gamma \approx 0$ in G then prove that $\frac{1}{2\pi i}\int_\gamma f = \sum_{k=1}^{n} \eta(\gamma_j a_k)$ Res

$(f_j a_k)$. **5**

c) Show that for a >1 $\int_0^\pi \frac{d\theta}{at\cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}$ **3**

8. a) Suppose f and g are meromorphic in a neighborhood $\overline{B}(a_j R)$ with no zeros or poles on the circle $\gamma = \{z : |z - a| = R\}$. If Z_f, Z_g (P_f, P_g) are the number of zeros (poles) of f and g inside γ counted according to their multiplicities and if $|f(z) + g(z)| < |f(z)| + |g(z)|$ on γ then prove that $Z_f - P_f = Z_g - P_g$. **5**

b) State and prove Schwarz's Lemma. **6**

c) Let $f(z) = \dfrac{1}{z(z-1)(z-2)}$ give the Laurent expansion of f(z) for annuli ann (0, 0,1). **5**

[3821] – 304

M.A./M.Sc. (Sem. – III) Examination, 2010
MATHEMATICS
MT – 704 : Measure and Integration (New)
(2008 Pattern) (Optional)

Time : 3 Hours   Max. Marks : 80

N.B. : *1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define a Measurable space and give two examples of it. **4**

b) If Ei are measurable sets with $\mu Ei < \infty$ and Ei $\supset$ Ei+1 then prove that

$$\mu(\bigcap_{i=1}^{\infty} Ei) = \lim_{n\to\infty} \mu En.$$ **6**

c) Show that $\mu(E_1 \Delta E_2) = 0$ implies $\mu E_1 = \mu E_2$ provided that E_1 and $E_2 \in \mathbb{B}$. **6**

2. a) Let f be an extended real valued function defined on X. Then prove that the following are equivalent.

i) $\{x : f(x) < \alpha\} \in \mathbb{B}$ for each α.

ii) $\{x : f(x) \leq \alpha\} \in \mathbb{B}$ for each α.

iii) $\{x : f(x) > \alpha\} \in \mathbb{B}$ for each α.

iv) $\{x : f(x) \geq \alpha\} \in \mathbb{B}$ for each α. **8**

b) If μ is a complete measure and f is a measurable function then prove that f = g a.e implies g is measurable. **8**

3. a) Let a $ b are positive integers and f, g are measurable simple function. Then prove that $\int_E (af + bg)d\mu = a\int_E fd\mu + b\int_E gd\mu$. **4**

b) State and prove Fatou's Lemma. **8**

c) Let $(X, \mathbb{B}, \mu)$ be a measure space and g be a nonnegative measurable function on X. Let $\gamma : \mathbb{B} \to [0,\infty]$ s.t $\gamma(\phi) = 0$. If $\gamma(E) = \int_E g\,d\mu$ then show that γ is a measure on $\mathbb{B}$. **4**

P.T.O.

4. a) Let g be integrable over E and suppose that <fn> is a sequence of measurable functions such that $|fn(x)| \leq g(x)$ and almost everywhere on E and $fn(x) \to f(x)$. Then prove that $\int_E f = \lim \int_E fn$. **6**

b) Prove that a measurable set A is a null set if and only if every measurable subset of A has signed measure zero. **4**

c) Let γ be a signed measure on $(X, \mathbb{B})$. Prove that there is a positive set A and a negative set B such that $X = A \cup B$ and $A \cap B = \phi$. **6**

5. a) State and prove Lebesgue Decomposition theorem. **8**

b) State and prove Riesz Representation theorem for a bounded linear functional on $\angle^p(\mu)$ with $1 \leq P < \infty$ and μ is a σ – finite measure. **8**

6. a) The class $\mathbb{B}$ of μ^* – measurable is a σ algebra. If $\bar{\mu}$ is μ^* restricted to $\mathbb{B}$ then prove that $\bar{\mu}$ is a complete measure on $\mathbb{B}$. **8**

b) Prove that set function μ^* is an outer measure and if $A \in a$ prove that A is measurable with respect to μ^*. **8**

7. a) If μ is a finite Baire measure on real line, then prove that its cumulative distribution function F is monotone increasing bounded function which is continuous from right and $\lim_{x \to -\infty} F(x) = 0$ **8**

b) Let $\{(Ai \times Bi)\}$ be a countable disjoint collection of measurable rectangle whose union is a measurable rectangle A × B prove that $\lambda(A \times B) = \sum \lambda(Ai \times Bi)$. **4**

c) State Fubini's Theorem. **4**

8. a) Let B be a μ^* measure set with $\mu^* B < \infty$ Prove that $\mu_* B = \mu^* B$. **8**

b) It μ^* is a Caratheodary outer measure with respect to set of real valued function Γ on any set X then prove that every function in Γ is μ^* measurable. **8**

[3821] – 304

M.A./M.Sc. (Sem. – III) Examination, 2010
MATHEMATICS
MT – 704 : Mathematical Methods – I (Old)
(Optional)

Time : 3 Hours Max. Marks : 80

N.B. : *1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Explain comparison test, integral test, ratio test for convergence of series of positive terms. **6**

b) Find whether the following series converges or diverges

$$\sum_{n=3}^{\infty} \frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}.$$ **5**

c) Find the interval of the convergence of the power series $\sum_{n=0}^{\infty} \frac{(2x)^n}{3^n}$ **5**

2. a) Find the first 4 terms of the Taylor series for, In X about X = 1. **4**

b) Find the Amplitude, Period, Frequency and wave velocity of the wave motion of particle whose distance 'S' from the origion is the function

$$S = \frac{1}{2} \cos(\pi t - 8).$$ **4**

c) Find the Fourier series expansion for the function $f(x) = \begin{cases} 0, -\pi < x < 0 \\ 1, 0 < x < \pi \end{cases}$. **8**

3. a) Find the Fourier coefficient with usual notation. **8**

b) Prove that, if f(x) has period P, average value of f is the same over any interval of length P. **8**

4. a) Define Gamma function and

Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ **8**

b) Show that B (p, q) = 2 $\int_0^{\pi/2} (\sin\theta)^{2p-1}(\cos\theta)^{2q-1}\,d\theta$ **8**

5. a) Define Legendre form of eliptic integral of first and second kind and find the arclength of an ellipse. **8**

b) Define Stirling formula and

evaluate $\lim_{n\to\infty} \frac{\Gamma\left(n+\frac{3}{2}\right)}{\sqrt{n}\cdot\Gamma(n+1)}$ **8**

6. a) Define Rodrigues formula and hence find $P_0(x)$, $P_1(x)$ and $P_2(x)$. **6**

b) Prove that $\int_{-1}^{1} P_m(x)P_n(x)\,dx = 0$ if m # n

Where Pn – denotes Legendre polynomial. **5**

c) Define $J_p(x)$ and prove that $\frac{d}{dx}J_0(x) = -J_1(x)$ with usual notation. **5**

7. a) Prove that $H'_n(x) = 2n\,H_{n-1}(x)$. **4**

b) Find Laplace transform of $(5 - 3e^{-2t})$. **4**

c) Use transform method to solve :

$y'' + 4y' + 13y = 20e^{-t}, y_0 = 1, y'_0 = 3.$ **8**

8. a) State and prove Parseval's theorem for fourier transform. **8**

b) Define laplace transform of dirac delta function and using laplace transform fo dirac delta function solve the following equation,

$$\frac{d^2y}{dt^2} + w^2y(t) = \delta(t - t0); y_0 = y'_0 = 0$$ **8**

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[3821] – 401

M.A./M.Sc. (Semester – IV) Examination, 2010
MATHEMATICS
MT – 801 : Field Theory (New)
(2008 Pattern)

Time: 3 Hours Max. Marks: 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Let $f(x) \in \mathbb{Z}[x]$ be a primitive polynomial. Show that f(x) is reducible over $\mathbb{Q}$ if and only if f(x) is reducible over $\mathbb{Z}$. **6**

   b) Let E be an extension field of F and $u \in E$ be algebraic over F. Let $p(x) \in F[x]$ be a polynomial of least degree such that p(u) = 0. Show that p(x) is an irreducible polynomial in F[x]. Further, show that if $g(x) \in F[x]$ such that g(u) = 0 then p(x) divides g(x). **6**

   c) Show that $x^3 + 3x + 2 \in \mathbb{Z}/(7)[x]$ is irreducible over the field $\mathbb{Z}/(7)$. **4**

2. a) Let F be a field. Show that there exists an algebraically closed field K containing F as a subfield. **10**

   b) Prove that $\sqrt{2}$ and $\sqrt{3}$ are algebraic over $\mathbb{Q}$, further, show that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right) = \mathbb{Q}\left(\sqrt{2} + \sqrt{3}\right)$. Hence find the degree of $\mathbb{Q}\left(\sqrt{2} + \sqrt{3}\right)$ over $\mathbb{Q}$. **6**

3. a) Let K be a splitting field of the polynomial $f(x) \in F[x]$ over the field F. If E is another splitting field of f(x) over F, then show that there is an isomorphism $\sigma : E \to K$ that is identity on F. **6**

   b) Let E be an extension of F of degree 2. Show that E is a normal extension. **4**

   c) Find the splitting field of $x^3 - 2 \in \mathbb{Q}[x]$ and also find the degree of the splitting field. **6**

4. a) If E is a finite separable extension of a field F, then prove that E is a simple extension of F. **7**

   b) If α is algebraic over a field F then $F[\alpha] = F(\alpha)$. **5**

   c) Prove that a finite extension of a finite field is a separable extension. **4**

P.T.O.

5. a) Let F and E be fields, and let $\sigma_1,........,\sigma_n$ be distinct embedding of F into E. Suppose that, for $a_1,......,a_n \in E$, $\sum_{i=1}^{n} a_i \sigma_i (a) = 0$ for all a ∈ F. Then prove that $a_i = 0$ for $1 \leq i \leq n$. **6**

b) Prove that every element of a finite field can be written as sum of two squares. **5**

c) Let F be a field with four elements. Find irreducible polynomials over F of degrees 2, 3 and 4. **5**

6. a) Let E be a Galois extension of F. Let K be any subfield of E containing F. Show that K is a normal extension of F if and only if G (E/K) is a normal subgroup of G(E/F). **8**

b) Find the splitting field of the polynomial $x^4 - 2 \in \mathbb{Q}[x]$ and determine its Galois group. **8**

7. a) Let f(x) ∈ F [x] have r distinct roots in the splitting field E over F. Show that the Galois group G(E/F) of f(x) is a subgroup of the symmetric group S_r. **6**

b) Show that the polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals over $\mathbb{Q}$. **7**

c) '$\mathbb{C}$ is algebraically closed. $\mathbb{Q}$ is a subfield of $\mathbb{C}$. Hence, $\mathbb{C}$ is algebraic closure of $\mathbb{Q}$. Is this argument correct ? Justify. **3**

8. a) If a > 0 is costructible number, then show that $\sqrt{a}$ is constructible. **6**

b) Show that the angle of measure 60° can not be trisected by ruler and compass. **6**

c) Determine the minimal polynomial of $\sqrt{-1+\sqrt{2}}$ over $\mathbb{Q}$. **4**

[3821] – 401

M.A./M.Sc. (Semester – IV) Examination, 2010
MATHEMATICS
MT – 801 : Algebraic Topology (Old Course)

Time: 3 Hours Max. Marks: 80

N.B. *: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Prove that the path homotopy relation is an equivalence relation. **6**

b) Show that if h, h' : X $\to$ Y are homotopic and k, k' : Y $\to$ Z are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic. **5**

c) Let r : X $\to$ A be a retraction map. If $a_0 \in A$, show that the map $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ defined by $r_*([f]) = [r \circ f]$ is surjective. **5**

2. a) Prove that the map P : $\mathbb{R} \to S^1$ given by the equation p(x) = (cos 2πx, sin 2πx) is a covering map. **6**

b) Let P : E $\to$ B be continuous and surjective. Suppose that U is an open set of B that is evenly covered by P. Show that if U is connected, then the partition of P^{-1} (U) into slices is unique. **5**

c) Let P : E $\to$ B be a covering map, let $P(e_0) = b_0$. Show that if E is path connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \to P^{-1}(b_0)$ is surjective. Further, show that if E is simply connected, then ϕ is bijective. **5**

3. a) State Brouwer fixed-point theorem for the disc. Prove it by using the theory of fundamental groups. **6**

b) Show that if h : $s' \to s'$ is nullhomotopic, then h has a fixed point and h maps some point x to its antipode –x. **5**

c) Given a polynomial equation $x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1x + a_0 = 0$ with real or complex coefficients. Show that if $|a_{n-1}| + \ldots\ldots + |a_1| + |a_0| < 1$, then all the roots of the equation lie interior to the unit ball B^2. **5**

4. a) Given two bounded polygonal regions in $\mathbb{R}^2$, prove that there exists a line in $\mathbb{R}^2$ that bisects each of them. **6**

b) Show that if g : $S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective. **5**

c) Let f : X $\to$ Y be continuous, let $f(x_0) = y_0$. If f is a homotopy equivalence, then prove that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. **5**

5. a) If $n \geq 2$, show that the n-sphere S^n is simply connected. **6**

b) Let X be the quotient space obtained from B^2 by identifying each point x of S' with its antipode –x. Show that X is homeomorphic to the projective plane P^2. **5**

c) Show that the fundamental group of the double torus is not abelian. **5**

6. a) If U is an open subset of $\mathbb{R}^2$ and $f : U \rightarrow \mathbb{R}^2$ is continuous and injective, then prove that f(U) is open in $\mathbb{R}^2$ and the inverse function $f^{-1} : f(U) \rightarrow U$ is continuous. **6**

b) Give examples to show that a simple closed curve in the torus may or may not separate the torus. **5**

c) Show that the fundamental group of theta-space is a free group on two generators. **5**

7. a) Let X be the space obtained from a finite collection of polygonal regions by pasting edges together according to some labelling scheme. Prove that X is a compact Hausdorff space. **6**

b) If X is the m-fold connected sum of projective planes, then prove that the torsion subgroup T(X) of $H_1(X)$ has order 2, and $H_1(X)/T(X)$ is a free abelian group of rank m-1. **5**

c) Let w be a proper scheme of the form $w = w_0 \, (cc) \, (aba^{-1}b^{-1}) \, w_1$. Prove that w is equivalent to the scheme $w_1 = w_0 \, (aabbcc) w_1$. **5**

8. a) Let $P : E \rightarrow B$ and $P' : E' \rightarrow B$ be covering maps, let $P(e_0) = P'(e_0') = b_0$. Prove that there is an equivalence $h : E \rightarrow E'$ such that $h(e_0) = e_0'$ if and only if the groups $H_0 = P_*(\pi_1(E, e_0))$ and $H_0' = P_*'(\pi_1(E', e_0'))$ are equal. Show that if h exists, then it is unique. **6**

b) Show that if $n > 1$, every continuous map $f : S^n \rightarrow S'$ is nulhomotopic. **5**

c) Let $P : E \rightarrow B$ be a covering map, let $P(e_0) = b_0$. If E is simply connected, then prove that b_0 has a neighbourhood U such that the inclusion $i : U \rightarrow B$ induces the trivial homomorphism $i_* : \pi_1(U, b_0) \rightarrow \pi_1(B, b_0)$. **5**

M.A./M.Sc. (Sem. – IV) Examination, 2010
MATHEMATICS
MT – 802 : Combinatorics (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : *1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) How many ways are there to pick two different cards from a standard 52-card deck such that :

   i) The first card is an Ace and the second card is not a queen ?

   ii) The first card is a spade and the second card is not a queen ? **6**

   b) How many four-digit numbers are there formed from the digits 1, 2, 3, 4, 5 (with possible repetition) that are divisible by 4 ? **4**

   c) How many 5-letter sequences (formed from the 26 letters in the alphabet, with repetition allowed) contain exactly one A and exactly two B's ? **6**

2. a) What is the probability of randomly choosing a permutation of the 10 digits 0, 1, ..., 9 in which an odd digit is in the first position and 1, 2, 3, 4 or 5 is in the last position ? **5**

   b) How many arrangements of the letters in MATHEMATICS are there in which TH occur together but the TH is not immediately followed by an E ? **6**

   c) How many ways are there to distribute three different chocolates and nine identical oranges to four children ? **5**

3. a) How many integer solutions are there to the equation
   $x_1 + x_2 + x_3 + x_4 = 12$,
   with $x_i \geq 0$? How many solutions
   with $x_1 \geq 2$, $x_2 \geq 2$, $x_3 \geq 4$, $x_4 \geq 0$? **6**

P.T.O.

b) Give a combinatorial argument to show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots\ldots\ldots + \binom{n}{n}^2 = \binom{2n}{n}$$ **5**

c) Find a generating function for a_r, the number of ways to distribute π identical objects into 5 distinct boxes with an even number of objects not exceeding 10 in the first two boxes and between three and five in the other boxes. **5**

4\. a) Find the coefficient of x^{16} in $(x^2 + x^3 + \text{---})^5$. What is the coefficient of x^r ? **6**

b) Use a generating function for modelling the number of distributions of 18 chocolate bunny rabbits into four Easter baskets with at least three rabbits in each basket. Which coefficient do we want ? **5**

c) Find a generating function for a_r, the number of ways that we can choose 2₵ A3₵and 5₵ stamps adding to a net value of r ₵. **5**

5\. a) Find the number of ways to place 25 people into three rooms with atleast one person in each room. **6**

b) Find a recurrence relation for a_n, the number of n – digit ternary sequences without any occurrence of the subsequence "012". **6**

c) Find a recurrence relation for the number of ways to distribute n distinct objects into five boxes. What is the initial condition ? **4**

6\. a) An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs. Find a recurrence relation and a formula for a_n, the number of different ways for the elf to ascend the n – stair stair case. **6**

b) Solve the recurrence relation

$a_n = 2a_{n-1} + 3a_{n-2}$ with $a_0 = a_1 = 1$ **5**

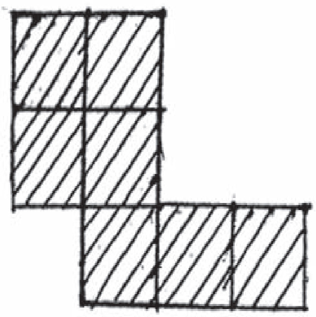
c) Solve the recurrence relation $a_n = 2a_{n-1} + 1$ with $a_1 = 1$. **5**

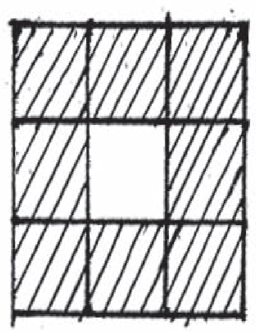
7\. a) How many n-digit ternary (0, 1, 2) sequences are there with atleast one 0, atleast one 1, and atleast one 2 ? **6**

b) How many ways are there to distribute r distinct objects into five (distinct) boxes with at least one empty box ? **5**

c) A wizard has five friends. During a long wizards' conference, it met any given friend at dinner 10 times, any given pair of friends 5 times, any given threesome of friends 3 times, any given foursome 2 times, and all 5 friends together once. If in addition it ate alone 6 times, determine how many days the wizards' conference lasted. **5**

8. a) Find the rook polynomial for the following boards : **8**

i) 

ii) 

b) State and prove the Inclusion – Exclusion formula. **8**

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M.A./M.Sc. (Sem. – IV) Examination, 2010
MATHEMATICS
MT – 802 : Hydrodynamics (Old Course)

Time : 3 Hours Max. Marks : 80

N.B. : *1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Derive the equation of continuity in Eulerian form. **6**

b) Explain the method of differentiation following the fluid motion. Obtain an analytic expression for the acceleration **6**

c) Find the stream lines for the fluids with velocity potential $\varphi = \lambda \times \gamma$ where λ is a constant. **4**

2. a) Obtain the Bernoulli's equation for the unsteady irrotational motion of an incompressible liquid. What is its form when motion is steady ? **8**

b) Prove that any relation of the form w = f(z) where $w = \varphi + i\psi$ and z = x + iy and f is analytic function ; represents a two dimensional irrotational motion in which the magnitude of velocity is given by $\left|\frac{dw}{dz}\right|$ **8**

3. a) Show that if vorticity is zero at any instant t_o, then it remains zero at all future times. **8**

b) In a cylindrical coordinate system, (r, θ, z) the radial component of velocity $\overline{q} = (q_r, q_\theta)$ of a two dimensional irrotational flow $q_r = \frac{3}{2} Ur^{3/2} \cos \frac{3\theta}{2}$. Find the expression for q_θ, if $q_\theta = 0$ at $\theta = 0$. Obtain the equation of stream lines given that the stream function $\Psi = 0$ at $\theta = \frac{2\pi}{3}$. **8**

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M.A./M.Sc. (Sem. – I)Examination, 2010
MATHEMATICS
MT-504 : Number Theory (New Course)
(2008 Pattern)

Time: 3 Hours Max. Marks : 80

N.B. : *1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1\. a) State and prove the fundamental theorem of arithmetic. **7**

b) If (a, m) = (b, m) = 1, then prove that (ab, m) = 1. **4**

c) If $2^n + 1$ is an odd prime for some integer n, prove that n is a power of 2. **5**

2\. a) State and prove Wilsons Theorem. **7**

b) Prove that $\frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is an integer for every $n \in \mathbb{N}$. **5**

c) Show that $1^2, 2^2, \ldots, m^2$ is not a complete residue system modulo m if m > 2. **4**

3\. a) Let a, b and m > 0 be given integers, and put g = (a, m). Prove that the congruence $ax \equiv b \bmod m$ has a solution if and only if g|b. Also prove that if this condition is met, then the solutions form an arithmetic progression with common difference $\frac{m}{g}$, giving g solutions modulo m. **8**

b) Solve the set of congruences :
$x \equiv 1 \pmod 4$
$x \equiv 0 \pmod 3$
$x \equiv 5 \pmod 7$. **4**

c) Characterize the set of positive integers n satisfying $\phi(2n) = \phi(n)$. **4**

4\. a) Let f(n) be a multiplicative function and let $F(n) = \sum_{d|n} f(d)$. Then prove that F(n) is multiplicative. **6**

b) Prove that $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$ if m is a positive integer. **4**

c) Prove that the number of divisors of n is odd if and only if n is a perfect square. **6**

P.T.O.

5. a) For any odd prime p let (a, p) = 1. Consider the integers a, 2a, 3a, . . . , $\left\{{(p-1)}/{2}\right\}$ a and their least positive residues modulo p. If n denotes the number of these residues that exceed $p/2$ then prove that $\left(\frac{a}{p}\right) = (-1)^n$. **6**

b) Find all odd primes p such that 3 is a quadratic residue modulo p. **6**

c) Suppose that p is an odd prime. Let n denote the least positive quadratic non residue modulo p. Then prove that $n < 1 + \sqrt{p}$. **4**

6. a) If an irreducible polynomial p(x) divides the product f(x) g(x) then prove that p(x) divides atleast one of the polynomials f(x) and g(x). **5**

b) Prove that the minimal equation of an algebraic integer is monic with integral coefficients. **5**

c) If a polynomial f(x) with integral coefficients factors into a product g(x) h(x) of two polynomials with coefficients in $\mathbb{Q}$, prove that there is a factoring $g_1(x)\, h_1(x)$ with integral coefficients. **6**

7. a) Define a prime in the quadratic field $Q(\sqrt{m})$. If the norm of an integer α in $Q(\sqrt{m})$ is $\pm p$, where p is a rational prime, then prove that α is a prime. **6**

b) Prove that if α is algebraic of degree n, then $\alpha - 1$ is also algebraic of degree n. **5**

c) Prove that $\sqrt{3} - 1$ and $\sqrt{3} + 1$ are associates in $Q(\sqrt{3})$. **5**

8. a) Let m be a square free rational integer positive or negative, but not equal to 1. Prove that the numbers of the form $a + b\sqrt{m}$ with rational integers a and b are integers of $Q(\sqrt{m})$. Also prove that these are the only integers of $Q(\sqrt{m})$ if (m) $\equiv$ 2 or 3 (mod 4) and if m $\equiv$ 1 (mod 4) then the numbers $\frac{a+b\sqrt{m}}{2}$ with odd rational integers a and b are also integers of $Q(\sqrt{m})$, and there are no further integers. **10**

b) Prove that 3 is a prime in Q(i), but not a prime in $Q(\sqrt{6})$. **6**

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M.A./M.Sc. (Semester – II) Examination, 2010
MATHEMATICS
MT-605 : Partial Differential Equations (New)
(2008 Pattern)

Time: 3 Hours Max. Marks : 80

N.B. *: i) Attempt* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Obtain the partial differential equation by eliminating arbitrary constants a and b from $Z = (x^2 + a)(y^2 + b)$ **4**

b) Find the general solution of : **4**

$xp + yq = z$

c) Find the complete integral of $z^2 (p^2z^2 + q^2) = 1$ by Charpits method. **8**

2. a) Explain the method of solving the following first order partial differential equations :

i) $f(z, p, q) = 0$ ii) $g(x, p) = h(y, q)$. **8**

b) Find the integral surface of the equation $x^3p + y(3x^2 + y)q = z(2x^2 + y)$ which passes through the curve $C : x_0 = 1, y_0 = S, z_0 = S(1 + S)$. **8**

3. a) Prove that : If $h_1 = 0$ and $h_2 = 0$ are compatible with $f = 0$ then h_1 and h_2 satisfy

$$\frac{\partial (f, h)}{\partial (x, u_x)} = \frac{\partial (f, h)}{\partial (y, u_y)} = \frac{\partial (f, h)}{\partial (z, u_z)} = 0$$

Where $h = h_i$ (i = 1, 2). **6**

b) Solve the initial value problem for the quasi linear equation $z_x - zz_y + z = 0$, with initial data curve $C : x_0 = 0, y_0 = S, z_0 = -2S, -\infty < S < \infty$. **6**

c) Show that the equations

$f = xp - yq - x = 0$

$g = x^2p + q - xz = 0$

are compatible find a one parameter family of common solutions. **4**

4. a) Reduce the equation,

$$y^2u_{xx} - 2xy\, u_{xy} + x^2\, u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y,$$ into a canonical form and solve it. **8**

b) Find by the method of characteristics, the integral surface of pq = xy which passes through the curve z = x, y = 0. **8**

5. a) If u (x, y) is harmonic in a bounded domain D and continuous in $\overline{D} = D \cup B$, prove that u attains its minimum on the boundary B of D. **4**

b) Prove that the solution of the Neumann problem is unique up to the addition of a constant. **4**

c) Find the solution of the one - dimensional heat equation :

$u_t = \alpha^2\, u_{xx}$ (0 < x < 1) (t > 0) with conditions :
$u(x, 0) = f(x) \quad 0 \leq x \leq 1$
$u(0, t) = u(1, t) = 0,\ t \geq 0.$ **8**

6. a) Show that the solution of following problem, if it exists then it is unique : **8**
$u_{tt} - c^2u_{xx} = F(x, t),\ 0 < x < a$
$u(x, 0) = f(x)\ 0 \leq x \leq a$
$u_t(x, 0) = g(x)\ 0 \leq x \leq a$
$u(0, t) = u(a, t) = 0,\ t \geq 0.$

b) State Dirichlet problem for rectangle and find it's solution. **8**

7. a) State and prove Kelvin's inversion theorem. **8**

b) Prove that for the equation $u_{xx} + \frac{1}{4}u = 0$; the Riemann function is $V(x, y, \alpha, \beta) = J_0\left(\sqrt{(x-\alpha)(y-\beta)}\right)$.

Where J_0 denotes the Bessel's function of the first kind of order zero. **5**

c) Find a solution of the partial differential equation $u_{xx} = x + y$. **3**

8. a) State and prove Harnack's theorem. **8**

b) Classify the following equations into hyperbolic, parabolic or elliptic type :
i) $u_{xx} + 2\, u_{yz} + \cos_x u_z = e^{y^2} u + \cosh z$
ii) $u_{xx} + 2(1 + \alpha y)\, u_{yz} = 0.$ **(4+4)**

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M.A./M.Sc. (Sem. – II) Examination, 2010
MATHEMATICS
MT - 606 : Object Oriented Programming Using C^{++} (New Course)
(2008 Pattern)

Time : 2 Hours　　　　　　　　　　　　　　　　Total Marks : 50

N.B. : *i) Question **1** is **compulsory**.*
*ii) Attempt **any two** from questions 2, 3, 4.*
*iii) Figures to the **right** indicate **full** marks.*

1. Attempt the following questions : **20**

   i) Write a short note on function prototype.

   ii) What will be the O/P of the following program

```
main ( )
 {
   int  i = 0,
   for ( ; i < = 10 ; ++i)
   Cout < < i;
 }
```

   iii) Write a function to read a matrix of size 5×6 from the keyboard using "for" loop.

   iv) Identify the error in the following program

```
 # include < iostream.h >

Void main ( )

{

  int num [ ] = { 1, 2, 3, 4, 5, 6 };

  num[ 1 ] = = [ 1] num ? Cout < < "Success" : Cout < < " Error" ;

}
```

   v) Write a program to multiply and divide two real numbers a = 2.5 and b = 1.5 using inline functions.

vi) When do we need to use default arguments in a function ?

vii) Write four special characteristics of friend function.

viii) Give C^{++} syntax of a class which has private and public members.

ix) Give C^{++} syntax of

a) Pointers to objects

b) Pointers to function

x) What is operator overloading ?

2. i) Write a program to find volume of cube of side s, cylinder of radius r and height h and a rectangular box of length *l*, breadth b and height h. Using function overloading. (Enter the values of r, s, h, *l* & b from keyboard). **10**

ii) Write a note on inline functions. **5**

3. i) Write a program to perform the addition of time in the hour and minutes format. **8**

ii) Write a note on constructor and give an example of a constructor. **7**

4. i) Write a program to multiply two numbers m and n from keyboard using multiple inheritance. Define class M to entre value of m, class N to entre value of n and class p which inherit the classes M and N which display the result. **10**

ii) Write benefits of OOP (Object Oriented Programming. **5**

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[3821] – 302

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS
MT - 702 : Ring Theory (New)
(2008 Pattern)

Time : 3 Hours Total Marks : 80

N.B. : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) Define a Boolean ring. Prove that the only Boolean ring that is an integral domain is $\frac{Z}{2Z}$. **5**

b) Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in Z \right\}$ be the subring of $M_2(z)$ of upper triangular matrices. Prove that the map $\phi : R \to Z \times Z$ defined by $\phi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (a, d)$ is a ring homomorphism and find its kernel.

Is ϕ one-one, onto ?

Is $\frac{R}{\text{Ker } \phi}$ an integral domain ? **6**

c) Prove or disprove. **5**

If M is an ideal such that $\frac{R}{M}$ is a field then M is a maximal ideal.

2. a) If A is a subring of a ring R and B is an ideal of the ring R then prove that $\frac{A+B}{B} \cong \frac{A}{A \cap B}$. **6**

P.T.O.

b) Let $f(x) = x^2 + x + 1$ be an element of the polynomial ring $E = \mathbb{F}_2[x]$, F_2 is field with two elements. Find all elements of $\overline{E} = \frac{F_2[x]}{(f(x))}$ and show that $\overline{E}$ is field. **5**

c) Show that the quotient field of Z[i] is Q[i]. **5**

3. a) Prove that every ideal in a Euclidean domain is principal. **6**

b) Prove that the quadratic integer ring $Z[\sqrt{-5}]$ is not a Euclidean domain. **6**

c) State whether the following statements are **true** or **false** with justification. **4**

i) Subring of an Euclidean domain is a Euclidean domain.

ii) Quotient ring of an Euclidean domain is an Euclidean domain.

4. a) If R is any commutative ring such that the polynomial ring R[x] is a principal ideal domain then prove that R is necessarily a field. **4**

b) Prove that in general a quotient of a P.I.D need not be a P.I.D but a quotient of a P.I.D. by a prime ideal is a P.I.D. **6**

c) Prove or disprove : **6**

i) Subring of a P.I.D. is P.I.D.

ii) Product of two P.I.D's is a P.I.D.

iii) Each non-zero prime ideal in a P.I.D is a maximal ideal.

5. a) Prove that every irreducible element in a U.F.D. is a prime. **5**

b) Show that the ring $Z[i\sqrt{3}] = \{a + ib\sqrt{3} \mid a, b \in Z\}$ is not a P.I.D. **5**

c) Prove that the ideals (x) and (x, y) are prime ideals in Q [x, y] but ideal (x, y) is not principal in Q [x, y]. **6**

6. a) Use Gauss lemma to prove that if the integral domain R is U.F.D. then R[x] is also U.F.D. **6**

b) Determine whether the following polynomials are irreducible in Z(x). **4**

i) $x^4 + 1$

ii) $x^3 - 3x - 1$.

c) If f(x) is a polynomial in F[x], F is a field, then prove that $\frac{F[x]}{(f(x))}$ is a field if and only if f(x) is irreducible. **6**

7. a) Define a torsion element of an R-module M. **6**

Show that if R is an integral domain then the set of all torsion elements, T or(M), of an R-module M is a submodule of M.

Give an example of a ring R and R-module M such that T or(M) is not a submodule.

b) Let F = $\mathbb{R}$ = set of all real NOS : and let V = $\mathbb{R}^2$ and T : V $\rightarrow$ V be a linear transformation, which is rotation clockwise about the origin by $\pi/2$ radians. Show that V and o are the only F[x] sub-modules for this T. **4**

c) If I is a right ideal of a ring R then, define annihilator of I in an R-module M as $\{m \in M \mid am = 0 \text{ for all } a \in I\}$ prove or disprove.

Annihilator of I in M is a submodule of M. **6**

If I = 2Z. Describe the annihilator of I in $M = \frac{Z}{24} \times \frac{Z}{15Z} \times \frac{Z}{50Z}$.

8. a) If N is a submodule of M and if both $\frac{M}{N}$ and N are finitely generated then prove that M is also f.g. **5**

b) Define an irreducible R - module. If R is a commutative ring, Then show that R-module M is irreducible iff $M \cong \frac{R}{I}$ as an R – module. Where I is maximal ideal of R. **6**

c) Give an example of a module which is

i) Free

ii) Not free, justify. **5**

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[3821] – 302

M.A./M.Sc. (Semester – III) Examination, 2010
MATHEMATICS
MT - 702 : Mechanics (Old Course)

Time : 3 Hours   Total Marks : 80

N.B. : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) Explain the following terms :

i) Degrees of freedom

ii) Cyclic coordinates. **4**

b) For a dynamical system consisting of N mass particles state the **4**

i) principle of virtual work

ii) D'Alembert's principle.

c) Derive Lagrange's equations of motion using Hamilton's principle. **8**

2. a) Let $L(q, \dot{q}, t) = \frac{m}{2}\left(\dot{q}^2 \sin^2 wt + q\dot{q}w \sin 2wt + q^2 w^2\right)$, be the Lagrangian of a particle.

Find the Hamiltonian and Hamilton's equations of motion. **6**

b) Given a velocity dependent central force $F = \frac{1}{r^2}\left(1 - \frac{(\dot{r}^2 - 2r\ddot{r})}{C^2}\right)$.

Write Hamiltonian and Hamilton's equations of motion. **6**

c) Find the Lagrangian of a simple pendulum. **4**

3. a) Two masses m_1 and m_2 ($m_1 > m_2$) are connected by an inextensible string of length *l* and the string passes over a weightless pulley. Find

i) Number of generalised coordinates and constraint equations.

ii) Show that the acceleration is

$$a = \frac{m_1 - m_2}{m_1 + m_2} g.$$ **6**

b) Find the kinetic energy of a free particle of mass m in 3-dimensions in terms of spherical polar coordinates r, θ, ψ. **4**

c) If the Hamiltonian H of the system is $H(q_1, q_2, p_1, p_2) = q_1p_1 - q_2p_2 - aq_1^2 - bq_2^2$

then show that q_1q_2 and $\frac{p_1 - aq_1}{q_2}$ are

constants of motion, where a, b are constants. **6**

4. a) Show that the constraint of rigidity is conservative. **4**

b) Show that if Hamiltonian H of a system does not depend on time explicitly then it represents total energy of the system. **6**

c) Find the extremal curve of the functional

$$\int_0^{\pi} (y')^2 - y^2 + 4y \cos x \, dx \quad y(o) = y(\pi) = 0.$$ **6**

5. a) Find the transformation generated by the generating function **6**

$$F_1(q, Q) = q\,Q - \frac{mw}{2} q^2 - \frac{Q^2}{4mw}.$$

b) Prove that the following transformation is canonical only if $\alpha = \frac{1}{2}, \beta = 2$

$$Q = q^{\alpha} \cos \beta p, \; P = q^{\alpha} \sin \beta p.$$ **6**

c) Given the canonical transformation

$$Q = \frac{1}{2}\left(q^2 + p^2\right), P = -\tan^{-1}\frac{q}{p}, \text{ find } [Q, P]_{q,p}$$ **4**

6. a) Prove that for a particle moving in a central force field the areal velocity is constant. **4**

b) Derive the following equation for the path of a particle in the central force field.

$$\frac{d^2u}{d\theta^2} + u = -\frac{f\left(\frac{1}{u}\right)}{mh^2u^2} .$$ **8**

c) State Kepler's laws of planetory motion. **4**

7. a) Show that Poisson brackets satisfy [uv, w] = [u, w] v + u [v, w]. **4**

b) Write Hamilton's equations in terms of Poisson brackets. **2**

c) Prove the Jacobi identity [u, [v, w]] + [v, [w, u]] + [w, [u, v] = 0. **10**

8. a) Write Hamilton Jacobi equation for harmonic oscillator and solve. **8**

b) Show that Poisson bracket is invariant under the canonical transforamtion. **8**

M.A./M.Sc. (Sem. – III) Examination, 2010
MATHEMATICS
MT-707 : Graph Theory (Old)
(2005 Pattern)

Time: 3 Hours Max. Marks : 80

N.B. : *1) Answer* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Prove that if a graph G is bipartite then every circuit in G has even length. **6**

   b) Show that the complete bipartite graph $K_{3,3}$ is nonplanar. **6**

   c) Prove that every connected planar graph with less than 12 vertices has a vertex of degree atmost 4. **4**

2. a) Show that if a graph is not connected then its complement must be connected. **4**

   b) Draw planar graphs with the following types of vertices, if possible : **4**

      i) K vertices of degree K, K = 1, 2, 3, 4, 5.

      ii) Two vertices of degree 3 and four vertices of degrees.

   c) Prove that an undirected multigraph has an Euler cycle if and only if it is connected and has all vertices of even degree. **8**

3. a) Prove that every tournament has a Hamilton path. **8**

   b) Prove that the vertices in a triangulation of a polygon can be 3 - colored. **8**

4. a) What is the chromatic polynomial of the circuit graph of length 4, C_4. **4**

   b) If 56 people sign up for a tennis tournament, how many matches will be played in the tournament ? **2**

   c) Prove that there are n^{n-2} different undirected trees on n labels. **10**

5. a) Let T be an m-ary tree of height h with l leaves. Prove that $l \leq m^h$, and if all leaves are at height h, $l = m^h$. **6**

   b) Let T be a tree with average degree a. Determine n(T) in terms of a. **4**

   c) Write a note on the traveling salesperson problem. **6**

P.T.O.

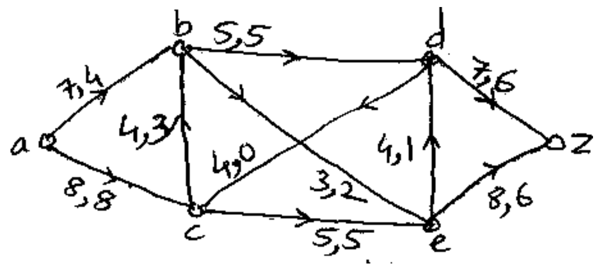
6\. a) Explain the 'shortest path algorithm'. **4**

b) Prove that the Prim's algorithm yields a minimal spanning tree. **8**

c) Prove that Kruskal's algorithm gives a minimal spanning tree. **4**

7\. a) Prove that for any a-z flow f and any a-z cut $(P, \overline{P})$ in a network N, $|f| \leq K(P, \overline{P})$. **6**

b) Apply the augmenting flow algorithm to the flow in following graph. **5**



c) Let G = (X, Y, E) be a bipartite graph and let N be the matching network associated with G. Prove that for any subsets $A \subseteq X$ and $B \subseteq Y$, $S = A \cup B$ is an edge cover if and only if $(P, \overline{P})$ is a finite capacity a-z cut in N, where $P = a \cup (X - A) \cup B$. **5**

8\. a) State and prove the 'Hall's Marriage Theorem'. **8**

b) Prove that for any a–z flow f(e) in a network N, the flow out of a equals the flow into Z. **4**

c) Prove by induction on m that Dijkstra's shortest path algorithm finds the shortest path from a to every other vertex in the network. **4**

M.A./M.Sc. (Semester – IV) Examination, 2010
MATHEMATICS
MT-804 : Algebraic Topology (New Course)
(2008 Pattern)

Time: 3 Hours Max. Marks : 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define homotopy between two continuous maps of a topological space X to another space Y. Illustrate by an example. Prove that homotopy is an equivalence relation on the set of all continuous maps of X to Y. **8**

b) Let $C_1 = \{x = (x_1, x_2) / (x_1 - 1)^2 + x_2^2 = 1\}$
$C_2 = \{x = (x_1, x_2) / (x_1 + 1)^2 + x_2^2 = 1\}$
and let $Y = C_1 \cup C_2$, $X = Y - \{(2, 0), (-2, 0)\}$ in $\mathbb{R}^2$.
Show that the point $x_0 = (0, 0)$ is a strong deformation retract of X. **4**

c) Show that the retract of a Hausdorff space is a closed set. **4**

2. a) Let $x_0, x_1 \in X$. If there is a path in X from x_0 to x_1 then prove that the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. **5**

b) Explain the terms :
i) Retract and retraction
ii) Deformation retract
iii) Strong deformation retract.

Bring out the relationships among them. Illustrate your answer with suitable examples. **7**

c) Let $\pi_1(X, x)$ be a trivial group. If f and g are two paths in X with $f(0) = g(0) = x$ and $f(1) = g(1)$, show that f and g are homotopic. **4**

3. a) Let σ and τ be paths at 1 in S' and $F : \sigma \simeq \tau$ (rel A). Prove that there is a unique continuous map $F' : C \times C \to R$ such that $\phi F' = F$ and $F' : \sigma' \simeq \tau'$(rel A). **6**

b) Show that the fundamental group $\pi_1(S')$ of the circle S' is isomorphic to the additive group $\mathbb{Z}$ of integers. **6**

c) Compute the fundamental group of the following spaces (if it exists) :
$\mathbb{Z}$, $\mathbb{R} \times \mathbb{R}$, $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, S^n (n > 1). **4**

4. a) Show that the circle S^1 is not a retract of the disc B^2. **6**

b) State fundamental theorem of algebra and give a proof using fundamental groups. **10**

5. a) Define : Covering projection and covering space. Give an example of a covering projection and an example which is not. Justify. **6**

b) Let X be a G-space. Show that the canonical projection $\pi: X \to X/G$ is an open mapping. **6**

c) Show that $\mathbb{R}/\mathbb{Z} \approx S'$. **4**

6. a) Let $P : \tilde{X} \to X$ be a covering map. Show that P is a fibration. **12**

b) Prove that the projection $P_2 : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is a fibration and for each $X \in \mathbb{R}^2$ the fiber $P_2^{-1}(x)$ is homeomorphic to $\mathbb{R}$. **4**

7. a) Explain terms with suitable examples : Geometrically independent set in $\mathbb{R}^n$, P-simplex, geometric simplicial complex in $\mathbb{R}^n$, polyhedron of a complex. **6**

b) If K is a complex and a barycentric subdivision K' of K exists, then prove that $|K| = |K'|$. **5**

c) Show that a simplex is convex. **5**

8. a) Write a short note on simplicial homology theory by explaining the terms : chains, cycles, boundaries and homology groups. **6**

b) Compute the simplicial homology groups of the standard 2-dimensional complex $K(s_2)$, where $S_2 = (a_0, a_1, a_2)$. **5**

c) Prove that the homomorphism :
$\phi_P : C_P(K) \to C_P(L)$ induces a homomorphism
$\phi_p^* : H_P(K) \to H_P(L)$. **5**

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M.A./M.Sc. (Semester – IV) Examination, 2010
MATHEMATICS
MT-804 : Mathematical Methods – II (Old Course)

Time: 3 Hours   Max. Marks : 80

N.B. *: 1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Define : **4**
   i) Symmetric Kernel
   ii) Fredholm integral equation of the first kind.

   b) Show that the function $y(s) = Sc^{s}$ is a solution of the equation **4**

$$y(s) = \sin s + 2\int_0^s \cos(s-t)\, y(t)\, dt .$$

   c) Explain the method to find the solution of integral equation **8**

$$F(s) = \phi(s) - \lambda\int_a^b K(s,t)\,\phi(t)\,dt$$

   When K(t, s) is degenerate.

2. a) Show that eigen functions of a symmetric Kernel corresponding to different eigen values are orthogonal. **8**

   b) Convert the initial value problem $y''(s) + sy = 1$ with $y(0) = y'(0) = 0$ to the volterra integral equation. **8**

3. a) Show that the Kernel

$$\sum_{n=1}^{\infty} \frac{\sin ns \sin nt}{n} = \frac{1}{2}\left|\sin\frac{s+t}{2}\right|\left|\sin\frac{s-t}{2}\right| (0 \le s, t \le \pi)$$

   has the eigen values $\lambda n = \frac{2n}{\pi}$ and the eigen functions sin nt. **8**

   b) Find the resultant Kernel of the volterra integral equation with the Kernel $K(x, t) = e^{x-t}$. **8**

4. a) Reduce the boundary value problem **8**

$$y''(s) + \lambda_P(s)\, y(s) = Q(s)$$

with y(a) = y(b) = 0 to a Fredholm integral equation.

b) Find the resolvent Kernel of the integral equation

$$y(s) = 1 + \lambda \int_0^1 (1 - 3st)\; y(t)\, dt.$$ **8**

5. a) State and prove the Euler - Lagrange's equation. **8**

b) Find the curve joining the points (x_1, y_1) and (x_2, y_2) that yields a surface of revolution of minimum area when revolved about the x-axis. **8**

6. a) State and prove isoperimetric problem. **8**

b) Derive Euler's equation for the functional **8**

$$I = \int_{x_0}^{x_1} f(x, y, y')\, dx \;;\; y(x_0) = a,\; y(x_1) = b$$

by using finite difference method.

7. a) State and prove the fundamental lemma of calculus of variation. **8**

b) Prove that, Geodesics on a sphere of radius 'a' is its great circle. **8**

8. a) State and prove Harr theorem. **8**

b) Find the stationary function of $\int_0^4 [xy' - (y')^2]\, dx$. **8**

Which is determined by the boundary conditions y(0) = 0 and y(4) = 3.

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M.A./M.Sc. (Sem. – IV) Examination, 2010
MATHEMATICS
MT – 807 : Combinatorics (Old Course)
(2005 Pattern)

Time: 3 Hours Max. Marks: 80

N.B. : *1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. A) How many arrangements of INSTRUCTOR are there in which there are exactly two consonants between successive vowels ? **6**

B) How many six letter sequences are there with at least three vowels (A, E, I, O, U) if no repetitions are allowed ? **6**

C) How many ways are there for Jack to pick an apple or a pear and then for Jill to pick an apple and a pear from 15 different apples and 10 different pears ? **4**

2. A) How many arrangements of the letters in PEPPERMILL are there with the first P appearing before the first L ? **6**

B) Prove by combinatorial argument : **6**

$C(r,r) + C(r+1, r) + C(r + 2, r) + ... + C(n, r) = C(n+1, r+1)$

Hence, evaluate the sum $1^2+2^2+3^2+ ......+ n^2$

C) How many nonnegative integer solutions are there to the equation : **4**

$2x_1 + 2x_2 + x_3 + x_4 = 12$?

3. A) Use generating functions to find the number of ways to select 300 chocolate candies from seven types if each type comes in boxes of 20 and if at least one but not more than five boxes of each type are chosen. **6**

B) How many r digit quaternary sequences are there in which the total number of 0's and 1's is even ? **6**

C) Using summation method, find a generating function with $a_r = 2r^2$. **4**

P.T.O.

4. A) Find recurrence relation for the number of n-digit ternary sequences with an even number of 0's and an even number of 1's. **6**

   B) Using generating functions, solve the recurrence relation : $a_n = 2a_{n-1} + 2^n$ Where $a_0 = 1$ **6**

   C) Solve the recurrence relation : **4**

   $a_n = 3a_{n-1} - 4n + 3(2^n)$, Where $a_1 = 8$

5. A) Solve the recurrence relation : **6**

   ${a_n}^2 = 2a^2_{n-1} + 1$ with $a_0 = 1$

   B) How many positive integers $\leq$ 30 are relatively prime to 30 ? **6**

   C) Among 40 toy robots, 28 have a broken wheel or are rusted, but not both. 6 are not defective and the number with a broken wheel equals the number with rust. How many robots are rusted ? **4**

6. A) State and prove inclusion-exclusion formula. **8**

   B) Seven DWarfs D_1, D_2, D_3, D_4, D_5, D_6, D_7 each must be assigned to one of seven jobs in a mine, J_1, J_2, J_3, J_4, J_5, J_6, J_7. If D_1 cannot do jobs J_1 or J_3 : D_2 cannot do J_1 or J_5 : D_4 cannot do J_3 or J_6 : D_5 cannot do J_2 or J_7 : D_7 cannot do J_4. D_3 and D_6 can do all jobs. How many ways are there to assign the dwarfs to different jobs ? **8**

7. A) State and prove Burnside's theorm. **8**

   B) A computer dating service wants to match four women each with one of five men. **8**

   If Woman 1 is incompatible with men 3 and 5 : Woman 2 is incompatible with men 1 and 2 : Women 3 is incompatible with man 4 : and Woman 4 is incompatible with men 2 and 4, how many matches of the four women are there ?

8. A) How many different necklaces can be made from beads of three colors black, white and red ? **6**

   B) Determine the pattern inventory for 3-bead necklaces distinct under rotations using black and white beads. **6**

   C) A baton is painted with equal sized cylindrical bands. Each band can be painted black or white. If the baton is unoriented, how many different 2 colorings of the baton are possible if the baton has 2 bands ? **4**

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