M.A./M.Sc. (Semester – I) Examination, 2011
MATHEMATICS
MT – 503 : Linear Algebra (2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. : i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) If V is a finite dimensional vector space over the field K and X, Y are finite subsets of V such that V= <X> and Y is linearly independent then prove that $|Y| \leq |X|$. **8**

   b) If $K_n [x]$ is a vector space, consisting of all polynomials in K[x] of degree at most n (K is a field) and if $W = \{p(x) \in K_n[x] / p(-x) = p(x), n \geq 2\}$ then show that W is a subspace of $K_n[x]$. What is the dim W ? For n = 7, find a basis consisting of $1 + x^2 + x^4$, $1 + x^4 + x^6$. **8**

2. a) If V is a finite dimensional vector space over the field K, and W is a subspace of V then prove that dim V = dim W + $\dim \frac{V}{W}$. **8**

   b) Let $T : \mathbb{R}[x] \to \mathbb{R}^2$ be a linear transformation defined by $T(p(x)) = \begin{bmatrix} p(1) \\ p(-1) \end{bmatrix}$ verify that T is a linear transformation. Find the kernel of T and determine into isomorphism $S : \frac{\mathbb{R}[x]}{\text{Ker}T} \to \mathbb{R}^2$. **8**

3. a) Let V and W be finite dimensional vector spaces over the field K and $T : V \to W$ be a linear transformation . With usual notation prove that
   Ker T* = (9m T)°
   and 9m T* = (Ker T)°
   What can you say about rank T and rank T* ? **8**

   b) Show that the set B = {$[2\ 1\ 1]^t$, $[3\ 4\ 1]^t$, $[2\ 2\ 1]^t$ } is a basis of the vector space $\mathbb{R}^3$.Find a basis of $(\mathbb{R}^3)^*$ dual to B. **8**

P.T.O.

4. a) If K is a field and $p(x) = x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_0$ is a monic polynomial of degree n in K[x]. Define the companion matrix associated with p(x).
If V is a K-vector space and T is a linear transformation on V, p(x) is a minimal polynomial of T of degree n, then prove that V has an ordered basis such that the matrix of T relative to this basis is the companion matrix C(p(x)). **8**

b) Find the rational canonical form of the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}$. **8**

5. a) Let T be a linear operator on an n-dimensional vector space V over the field K. If T has n-distinct eigen vectors then prove that T is diagonalizable. **8**

b) Show that 1 is an eigenvalue of the following matrix. **8**

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

Find geometric and algebraic multiplicities of 1.

6. a) i) Prove that in an inner product space an orthogonal set of non-zero vectors is linearly independent. **4**

ii) Prove that in a finite dimensional inner product space, a complete orthonormal set is a basis. **4**

b) If $V = \mathbb{R}_3[x]$ is an inner product space with inner product.

$(p(x), q(x)) = \int_{-1}^{1} p(x)q(x)dx$ and if $T : \mathbb{R}_3[x] \to \mathbb{R}$ given by T(p(x)) = p(0) is a linear transformation, then by Riesz representation theorem find $r(x) \in \mathbb{R}_3[x]$ such that T(p(x)) = (p(x), r(x)) for all $p(x) \in \mathbb{R}_3[x]$. **8**

7. a) Let V and W be finite dimensional inner product spaces and let $T \in L(V, W)$ prove that there exist a unique linear map $T^* : W \to V$ such that for all $v \in V$ and $w \in W$.

(Tv, w) = (v, Tw)

Also prove that T** = T. **8**

b) Prove that $\lambda \in K$ (K is a field) is eigenvalue of a linear transformation T if and only if $\overline{\lambda}$ is an eigenvalue of T*. **8**

8. a) If V is a finite dimensional vector space over the field K and if T is a linear operator on V.

Define T-cyclic subspace of V generated by $v \in V$. What is the basis of this subspace ? Show that a Jordan subspace for a linear operator T is T-cyclic. **8**

b) If A = $\begin{pmatrix} 1 & 0 & 2 \\ 3 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$ and if A has eigenvalue 0 with eigen vector $[2\ -1\ -1]^t$,

then find a unitary matrix U such that U*AU is upper triangular. **8**

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M.A. / M.Sc. (Semester – II) Examination, 2011
MATHEMATICS
MT – 601 : General Topology (New)

Time : 3 Hours Max. Marks : 80

N.B. : *1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let X be any non-empty set, $\mathcal{T}_c$ be the collection of all subsets U of X such that X – U is either countable or all of X. Is $\mathcal{T}_c$ a topology on X ? Verify. **5**

b) Let X be a topological space. Suppose that $\mathcal{C}$ is a collection of open sets of X such that, for each open set U of X and each x in U, there is an element C of $\mathcal{C}$ such that $x \in C \subseteq U$ then show that $\mathcal{C}$ is a basis for the topology of X. **6**

c) Define the lower limit topology and K-topology on $\mathbb{R}$ and show that both the topologies are not comparable. **5**

2. a) Define a subspace topology. If A is a subspace of $(X, \mathcal{T})$ and B is a subspace of $(Y, \mathcal{T}')$ then show that the product topology on A×B is same as the topology A×B inherits as a subspace of X × Y. **6**

b) Define projection maps and show that the collection $S = \{\pi_1^{-1}(U) / U \text{ open in } X\} \cup \{\pi_2^{-1}(V) / V \text{ open in } Y\}$ is a subbasis for the product topology on X × Y. **6**

c) Let Y = [–1, 1], which of the following sets are open in Y ? Open in $\mathbb{R}$? Justify

i) $A = \left\{ x \Big/ \frac{1}{2} \leq |x| < 1 \right\}$

ii) $B = \left\{ x \Big/ \frac{1}{2} < |x| < 1 \right\}$ **4**

3. a) Let Y be subspace of X and A be a subset of Y. Let $\overline{A}$ denote the closure of A in X then show that the closure of A in Y is $\overline{A} \cap Y$. **5**

b) Show that every finite set is closed in a Hausdorff space. **5**

c) Show that every order topology is Hausdorff. **6**

4. a) State and prove pasting lemma. **6**

b) Show that [0, 1] and [a, b] are homeomorphic. **6**

c) Let $\{ X_\alpha \}$ be an indexed family of topological spaces; let $A_\alpha \subset X_\alpha$ for each α. If ΠX_α is given either product or box topology then show that,

$\Pi \overline{A}_\alpha = \overline{\Pi A_\alpha}$. **4**

5. a) Show that the topologies on $\mathbb{R}^n$ induced by the Euclidean metrid d and the square matrix ρ are the same as the product topology on $\mathbb{R}^n$. **6**

b) State and prove sequence lemma. **5**

c) Prove that a restriction map is a quotient map. **5**

6. a) Prove that the image of connected space under a continuous map is connected. **5**

b) Prove that every compact subspace of a Hausdorff space is closed. **5**

c) Show that one-point compactification of Z+ is $\{0\} \cup \left\{ \frac{1}{n} \middle/ n \in z + \right\}$. **6**

7. a) Show that a subspace of a Hausdorff space is Hausdorff and a product of Hausdorff spaces is Hausdorff. **6**

b) Show that every compact, Hausdroff space is normal. **5**

c) With usual notations, show that R_l is Lindelöf space. **5**

8. a) State and prove Tychonoff theorem. **12**

b) State Urysohn lemma. **2**

c) Define a regular space and give its example. **2**

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M.A./M.Sc. (Semester – II) Examination, 2011
MATHEMATICS
MT-601 : Real Analysis – II (Old)

Time : 3 Hours Max. Marks : 80

N.B. : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) With usual notations, prove that $\|f_1 f_2\|BV \leq \|f_1\|BV\|f_2\|BV$. **5**

b) Let $f, g \in BV[a, b]$ and $a \leq c \leq b$, then prove that

$V_a^b(f + g) \leq V_a^b(f) + V_a^b(g)$ and

$V_a^b(f) = V_a^c(f) + V_c^b(f)$. **6**

c) Give example of a bounded function which is not Riemann integrable. Justify your answer. **5**

2. a) Show that the sum of two measurable functions is measurable. **6**

b) Let $f \in R_\alpha[a, b]$ and c be a real number. Prove that

$cf \in R_\alpha[a,b]$ and $\int_a^b cf\, d\alpha = c\int_a^b f d\alpha$. **5**

c) True or False ? Justify.
A bounded continuous function is of bounded variation. **5**

3. a) If $f \in R_\alpha[a, b]$, then prove that $\alpha \in R_f[a,b]$ and

$\int_a^b f d\alpha + \int_a^b \alpha df = f(b)\alpha(b) - f(a)\alpha(a)$. **6**

b) Write the Fourier series for the following function :

$f(x) = 0$ if $x \in [-\pi, 0)$

$= 1$ if $x \in [0, \pi)$. **6**

c) Define the outer measure. Give example of a set with outer measure zero. Justify. **4**

4. a) If E and F are disjoint compact sets, then prove that
$m^*(E \cup F) = m^*(E) + m^*(F)$. **6**

b) Show that the improper Riemann integral $\int_0^\infty \frac{\sin x}{x}\,dx$ exists. **6**

c) Let $\{f_n\}$ be a sequence of measurable functions, then show that $\sup_n f_n$ is measurable. **4**

5. a) If f is Riemann integrable on [a, b], then prove that f is Lebesgue integrable. **6**

b) Suppose f is a non-negative and measurable function, then show that $\int f = 0$ if and only if f = 0 a.e. **5**

c) If F is a closed subset of a bounded open set G, then prove that m*(G/F) = m* (G) – m*(F). **5**

6. a) If f is a measurable function, then show that | f | is measurable. Is the converse true ? Justify. **5**

b) Let $\{E_n\}$ be a sequence of measurable sets. If $E_n \supset E_{n+1}$ for each n and $m(E_k)$ is finite for some k, then prove that $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$. **6**

c) Show that $\lim_{n\to\infty}\int_0^1 f_n = 0$ where $f_n(x) = \frac{nx}{1+n^2x^2}$. **5**

7. a) State and prove Lebesgue's Monotone Convergence Theorem. **8**

b) Let $1 < p < \infty$ and q be defined by $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L_p(E)$ and $g \in L_q(E)$, then prove that $fg \in L_1(E)$ and $\left|\int_E fg\right| \le \int_E |fg| \le \|f\|_p \|g\|_q$. **5**

c) Define absolutely continuous function and give example. **3**

8. a) State and prove Fatou's Lemma. **8**

b) Give an example of a nonmeasurable set. **8**

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B/II/11/265

[4023] – 205

M.A./M.Sc. (Semester – II) Examination, 2011
MATHEMATICS
MT-605 : Partial Differential Equations
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : i) Attempt ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Obtain the D'Alemberts solution of
$C^2u_{xx} - u_{tt} = 0$;
$u(x, 0) = f(x),\ u_t(x, 0) = g(x)$ **8**

b) Find the general solution of : **4**

$xp + yq = z$

c) Eliminate Parameters a and b from the equation $x^2 + y^2 + (z - b)^2 = a^2$. **4**

2. a) Show that the following equations are compatible and find a one parameter family of their common solutions :

$f = xp - yq - x = 0$

$g = x^2p + q - xz = 0$ **4**

b) Solve the non-linear partial differential equation $p^2 + q^2 = 1$. **4**

c) If $h_1 = 0$ and $h_2 = 0$ are compatible with f = 0 then prove that h_1 and h_2 satisfy $\frac{\partial(f, h)}{\partial(x, u_x)} + \frac{\partial(f, h)}{\partial(y, u_y)} + \frac{\partial(f, h)}{\partial(z, u_z)} = 0$ **8**

3. a) Find the complete integral of px + qy = pq by Charpits method. **8**

b) Solve the quasi-linear equation $xz_y - yz_x = z$ with initial condition $z(x, 0) = f(x)$; $x \geq 0$. **6**

c) Derive Analytic expression for the Monge cone at (x_0, y_0, z_0). **2**

P.T.O.

4. a) Find characteristic strips of the equation $pq = xy$; and obtain a equation of the integral surface through the curve $z = x$, $y = 0$. **8**

b) Find the complete integral of a equation $p^2x + q^2y = z$ by Jacobi's method. **8**

5. a) Reduced the equation $(n-1)^2 u_{xx} - y^{2n} u_{yy} = n y^{2n-1} u_y$ to canonical form; where n is an integer. **6**

b) Prove that for the equation $Lu = u_{xy} + \frac{1}{4} u = 0$; the Riemann function is $V(x, y ; \alpha, \beta) = J_0\left(\sqrt{(x-\alpha)(y-\beta)}\right)$ where J_0 denotes the Bessel's function of the first kind of order zero. **6**

c) Find the solution of a partial differential equation $u_{yy} = x + y$. **4**

6. a) Using the variable separable method solve $u_{xx} + u_{yy} = 0$ which satisfies conditions $u(0, y) = u(l, y) = u(x, 0) = 0$ and $u(x, a) = \sin\left(\frac{n\pi x}{l}\right)$. **8**

b) If $u(x, y)$ is a harmonic function in a bounded domain D and continuous in $\overline{D} = D \cup B$ then prove that u attains it's maximum on the boundary B of D. **5**

c) Prove that the solution of the Neumann problem is unique up to the addition of a constant. **3**

7. a) State and prove Kelvin's inversion theorem. **8**

b) Find a solution of the heat equation in an infinite rod which is defined as

$u_t = k u_{xx}$; $-\infty < x < \infty$, $t > 0$

$u(x, 0) = f(x)$; $-\infty < x < \infty$ **8**

8. a) Explain the method of solving the following first order partial differential equations. **6**

i) $z = px + qy + g(p, q)$

ii) $g(x, p) = h(y, q)$

iii) $f(p, q) = 0$

b) Is the surface $x^2 + y^2 + z^2 = cx^{2/3}$ is equipotential. If yes then find potential function. **5**

c) Classify the following equations into hyperbolic, parabolic or elliptic type

i) $7u_{xx} - 10u_{xy} - 22u_{yx} + 7u_{yy} - 16u_{xz} - 5u_{zz} = 0$

ii) $u_{xx} + 2u_{xy} + u_{yy} + 2u_{zz} - (1 + xy)u = 0$ **5**

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B/II/11/250

M.A./M.Sc. (Semester – III) Examination, 2011
MATHEMATICS
MT-701 : Functional Analysis (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. : i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) Let M be a closed linear subspace of a normed linear space N. If a norm of a coset x + M in the quotient space N/M is defined by $|||x + M||| = \inf\{\|x+m\| : m \in M\}$, then prove that N/M is a normed linear space. Further if N is Banach, then prove that N/M is also a Banach space. **8**

   b) Show that $\|T^*\| = \|T\|$ and $\|T^*T\| = \|T\|^2$. **6**

   c) A linear operator $S : l^2 \to l^2$ is defined by $S(x_1, x_2, ...) = (0, x_1, x_2, ...)$. Find its adjoint S*. **2**

2. a) Let Y be a closed subspace of a normed linear space X. Show that a sequence $\{x_n + Y\}$ converges to x + Y in X/Y if and only if there is a sequence $\{y_n\}$ in Y such that $\{x_n + y_n\}$ converges to x in X. **6**

   b) State and prove Hahn-Banach Theorem. **8**

   c) Write example of a self-adjoint operator. **2**

3. a) Prove that a Banach space cannot have a denumerable Hamel basis. **6**

   b) Give examples of two non-equivalent norms. Justify. **6**

   c) If T is any operator on a Hilbert space H and if α, β are scalars such that $|\alpha| = |\beta|$, then show that $\alpha T + \beta T^*$ is normal. **4**

4. a) Show that an operator T on a Hilbert space H is normal if and only if its adjoint T* is a polynomial in T. **6**

   b) Let X and Y be normed spaces. If X is finite dimensional, then show that every linear transformation from X to Y is continuous. Give an example of a discontinuous linear transformation. **8**

   c) Show that the norm of an isometry is 1. **2**

5. a) If T is an operator on a Hilbert space H, then prove that T is normal if and only if its real and imaginary parts commute. **6**

b) Let M be a closed linear subspace of a normed linear space N and T be the natural mapping of N onto N/M defined by T(x) = x + M. Show that T is a continuous linear transformation for which $\|T\| \leq 1$. **6**

c) Find $M^{\perp}$ if $M = \{(x, y) : x - y = 0\} \subset R^2$. **4**

6. a) Show that the unitary operators on a Hilbert space H form a group. **4**

b) If T is an operator on a Hilbert space H for which $\langle Tx, x\rangle = 0$ for all $x \in H$, then prove that T = 0. **6**

c) Let S and T be normal operators on a Hilbert space H. If S commutes with T*, then prove that S+T and ST are normal. **6**

7. a) Let H be a Hilbert space and f be a functional on H. Prove that there exists a unique vector y in H such that $f(x) = \langle x, y\rangle$ for every $x \in H$. **8**

b) Let T be a normal operator on H with spectrum $\{\lambda_1, \lambda_2, ...\lambda_m\}$. Show that T is self-adjoint if and only if each λ_i is real. **4**

c) Let T be an operator on H. If T is non-singular, then show that $\lambda \in \sigma(T)$ if and only if $\lambda^{-1} \in \sigma(T^{-1})$. **4**

8. a) State and prove the Closed Graph Theorem. **8**

b) Prove that every finite dimensional subspace of a normed linear space X is closed. Give an example to show that an infinite dimensional subspace of a normed linear space may not be closed. **8**

M.A./M.Sc. (Semester – III) Examination, 2011
MATHEMATICS (Optional)
MT : 705 : Graph Theory (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. : 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define the Petersen graph. Prove that if two vertices are non-adjacent in the Petersen graph then they have exactly one common neighbor. **5**

   b) Prove that every set of six people contains (at least) three mutual acquaintances or three mutual strangers. **6**

   c) Prove that every u,v-walk contains a u,v-path. **5**

2. a) Prove that a graph is bipartite if and only if it has no odd cycle. **8**

   b) Prove that a graph is Eulerian if it has at most one nontrivial component and its vertices all have even degree. **8**

3. a) Let G be a simple graph with vertices $v_1, v_2, ..., v_n$, Let A^k denote the k^{th} power of the adjacency matrix of G under matrix multiplication. Prove that entry i, j of A^k is the number of v_i, v_j-walks of length k in G. **6**

   b) State and prove the Havel-Hakimi Theorem. **10**

4. a) Prove that the hypercube Q_k is k-regular and $e(Q_k) = k2^{k-1}$. **5**

   b) If T is a tree with k edges and G is a simple graph with $\delta(G) \geq k$, then prove that T is a subgraph of G. **5**

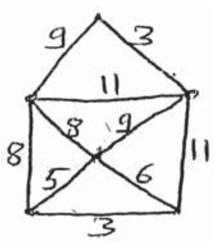
   c) Prove that every tree with at least two vertices has at least two leaves. Also prove that deleting a leaf from an n-vertex tree produces a tree with n – 1 vertices. **6**

5. a) Prove that among trees with n vertices, the Wiener index $D(T) = \sum_{u,v} d(u,v)$ is maximized by paths. **8**

b) Prove that for given positive integers $d_1, d_2, ..., d_n$ summing to 2n – 2, there are exactly $\frac{(n-2)!}{\pi(d_i - 1)!}$ trees with vertex set [n] such that vertex i has degree d_i, for each i. **8**

6. a) Let $\tau(G)$ denote the number of spanning trees of a graph G. Prove that if $e \in E(G)$ is not a loop, then $\tau(G) = \tau(G - e) + \tau(G.e)$. **8**

b) Using the Kruskal's Algorithm construct a minimum-weight spanning tree for the following graph. **5**



c) Show that a tree cannot have more than one perfect matching. **3**

7. a) Prove that a matching M in a graph G is a maximum matching in G if and only if G has no M-augmenting path. **8**

b) Explain the Augmenting path algorithm to produce a maximum matching. **4**

c) Prove that two blocks in a graph share atmost one vertex. **4**

8. a) Prove that a graph is 2-connected if and only if it has an ear decomposition. **8**

b) Prove that $\chi(G \cdot H) = \max\{\chi(G), \chi(H)\}$. **4**

c) Prove the Brooks Theorem when G is not k-regular. **4**

M.A./M.Sc. (Semester – IV) Examination, 2011
MATHEMATICS
MT : 804 : Algebraic Topology
(2008 Pattern) (New)

Time : 3 Hours Max. Marks : 80

*N.B. : 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Define homotopy between two continuous functions from X to Y. Prove that this relation is an equivalence relation. **6**

b) Let $S^1 \to X$ be a continuous map. Show that f is nullhomotopic if and only if there is a continuous map $g : B^2 \to X$ with $f = g|s^1$. **5**

c) Let $f_1 \approx f_2 : X \to Y$ and $g_1 \approx g_2 : Y \to Z$. Show that $g_1 \circ f_1 \approx g_2 \circ f_2 : X \to Z$. **5**

2. a) Define a contractible space. Show that unit n-ball B^n is contractible. Give an example of a noncontractible space. **6**

b) Define a deformation retract of space X. Prove that S^1 is a deformation retract of $\mathbb{R}^2 – \{0\}$. **5**

c) Show that a contractible space is path connected. **5**

3. a) Show that every open connected subset of $\mathbb{R}^2$ is path connected. **6**

b) Give an example of a connected subset of $\mathbb{R}^2$ which is not path connected. **5**

c) Let $A \subset B \subset X$. Suppose B is a retract of X, and A is a retract of B. Show that A is a retract of X. **5**

P.T.O.

4\. a) Let $x_0, x_1 \in X$. Show that if there is a path in X from x_0 to x_1 then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic. Hence prove that $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic for a connected path X with $x, y \in X$. **8**

b) Let $f : X \to Y$ is continuous. Show that there exists a homomorphism $f^* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ where x_0 is any point of X. **8**

5\. a) Prove that the fundamental group of the circle S^1 is isomorphic to the additive group $\mathbb{Z}$ of integers. **6**

b) Define a simply connected space. Prove that a contractible space is simply connected. Is converse true ? Justify. **5**

c) Prove that the retract of a Housdorff space is closed. **5**

6\. a) Define a covering map. Prove that $p : S^1 \to S^1$ defined by $p(z) = z^n$ is a projection map. Also prove that $\mathbb{R}$ is a covering space of S^1. **6**

b) Prove that a covering map is a local homeomorphism. Also show that the converse is not true. **5**

c) Prove that if $p_1 : X_1 \to Y_1$ and $p_2 : X_2 \to Y_2$ are covering maps then $p_1 \times p_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a covering map. **5**

7\. a) Prove that a fibration has unique path lifting if and only if every fibre has no non-null path. **8**

b) Suppose that K is a connected complex. Prove that the 0th homology group $H_0(K)$ of K is isomorphic with the additive group of integers. **8**

8\. a) Prove that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if n = m. **6**

b) Prove that two different complexes may have the same polyhedra. **5**

c) Prove that the surface of a sphere in $\mathbb{R}^3$ is a triangulable space. **5**

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M.A./M.Sc. (Semester – IV) Examination, 2011
MATHEMATICS
MT – 804 : Mathematical Method – II (Old Course)

Time : 3 Hours Max. Marks : 80

N.B. : *i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) Define :

i) Fredholm's integral equation of second kind

ii) Degenerate Kernels. **4**

b) Show that the function $u(s) = (1+s^2)^{-3/2}$ is a solution of the volterra integral equation $u(s) = \dfrac{1}{1+s^2} - \displaystyle\int_0^s \dfrac{t}{1+s^2}\, u(t)\,dt$. **6**

c) Reduce the following boundary value problem into an integral equation $\dfrac{d^2y}{dx^2} + \lambda y = x, \; y(0) = y(\pi) = 0$. **6**

2. a) Prove that, the eigenvalues of a symmetric kernel are real. **8**

b) Show that the homogeneous integral equation $\phi(x) - \lambda\displaystyle\int_0^1 (3x-2)\,\phi(t)\,dt = 0$ has no characteristic numbers and eigen functions. **8**

3. a) Find eigen values and eigen functions of the homogeneous Fredholm integral equation of the second kind $\phi(s) = \lambda \displaystyle\int_0^1 (2st - 4s^2)\,\phi(t)\,dt$. **8**

b) Solve the symmetric integral equation $y(x) = (x+1)^2 + \displaystyle\int_{-1}^{1} (xt + x^2t^2)\,y(t)\,dt$ by using Hilbert-Schmidt theorem. **8**

4. a) Find the iterated kernels for the kernel $k(s, t) = e^s \cos t \,;\, a = 0, b = \pi$. **8**

b) Find the resolvent kernel of the volterra integral equation with the kernel

$K(s, t) = \dfrac{2 + \cos(s)}{2 + \cos(t)}$. **8**

5. a) Find the Neumann series for the solution of the integral equation

$y(x) = 1 + x + \lambda \int_0^x (x - t)\, y(t)\, dt$. **8**

b) Let $\psi_1(s), \psi_2(s), \ldots$ be a sequence of functions whose norms are all below a fixed bound M and for which the relation $\psi_n(s) = \lambda \int K(s, t)\psi_n(t)\, dt$ holds in the sense of uniform convergence. Prove that, the functions $\psi_n(s)$ form a smooth sequence of functions with finite asymptotic dimension. **8**

6. a) State and prove isoperimetric problem. **8**

b) Find the extremal for $I = \int_0^{\pi/4} (y''^2 - y^2 + x^2)\, dx,\ y(0) = 0,\ y'(0) = 1,$

$y\left(\frac{\pi}{4}\right) = y'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. **8**

7. a) Find the shortest distance between circle $x^2 + y^2 = 4$ and straight line $2x + y = 6$. **8**

b) Evaluate the resolvent kernel of the integral $K(x, t) = 1 + 3xt\,;$ $0 \le x \le 1, 0 \le t \le 1$. **8**

8. a) State and prove Harr theorem. **8**

b) Show that an isosceles triangle has the smallest perimeter for a given area and given base (Use principle of reciprocity). **8**

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M.A./M.Sc. (Semester – I) Examination, 2011
MATHEMATICS
MT – 501 : Real Analysis – I
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : 1) Attempt ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Consider the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$.

   i) Prove that $\|x\| = \frac{1}{3}\|x\|_1 + \frac{2}{3}\|x\|_\infty$ defines a norm on $\mathbb{R}^n$.

   ii) Sketch the open unit ball in $\mathbb{R}^2$ with respect to this norm. **6**

   b) Show that the supremum norm on C([a, b]) can not come from an inner product. **5**

   c) Find the set of limit points of

   i) $\left\{\frac{1}{n} \mid n = 1, 2, 3, \ldots\right\} \cup \{0\}$ in $\mathbb{R}$

   ii) {1, 2, 3, ..., 100} in $\mathbb{R}$. **5**

2. a) Consider a metric space M and assume that $A \subseteq M$ is compact and that $f : M \to \mathbb{R}$ is continuous. Show that f(A) is a compact subset of $\mathbb{R}$. **6**

   b) Show that $(C([a, b]), \|\cdot\|_2)$ is not complete. **5**

   c) Show that the closed unit ball in l^1 is not compact. **5**

3. a) Let m be the Lebesgue measure defined on $\mathbb{R}^n$. Let $\mathscr{E}$ be the collection of all finite unions of disjoint intervals in $\mathbb{R}^n$. Prove that m is a measure on $\mathscr{E}$. **6**

   b) Let $\mathscr{R}$ be a σ-ring. Prove that $\bigcap_{n=1}^{\infty} A_n \in \mathscr{R}$ whenever $A_n \in \mathscr{R}$, n = 1, 2, ... **5**

   c) If f is a measurable function, then prove that | f | is also measurable. Is converse true ? Justify your answer. **5**

4. a) Assume that $f \geq 0$ is a measurable function and that $A_1, A_2, \ldots$ are pairwise disjoint measurable sets. Prove that $\int\limits_{\bigcup_{k=1}^{\infty} A_k} f\,dm = \sum\limits_{k=1}^{\infty} \left(\int\limits_{A_k} f\,dm \right)$. **7**

b) Prove that every continuous real valued function f defined on $\mathbb{R}^n$ is measurable. **5**

c) If f and g are Lebesgue integrable functions, then prove that f and g are equal almost everywhere if and only if $\int\limits_E f\,dm = \int\limits_E g\,dm$. **4**

5. a) State and prove Lebesgue monotone convergence theorem. **6**

b) True or false ?

If $\{f_n\}_{n=1}^{\infty}$ is a sequence of Riemann integrable functions on [a, b] with $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in [a, b]$. Then f is Riemann integrable and $\lim\limits_{n\to\infty} \int\limits_a^b f_n(x)dx = \int\limits_a^b f(x)dx$. Justify your answer. **6**

c) Give an example to show that strict inequality can hold in Fatou's lemma. **4**

6. a) State and prove Hölder's inequality. **5**

b) State and prove Bessel's inequality. **7**

c) Give an example to show that pointwise convergence does not imply convergence in mean. **4**

7. a) Show that the classical Fourier of $f(x) = x$ is $2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$. **5**

b) Use your work in (a), together with Parseval's identity, to obtain Euler's identity

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$ **5**

c) Obtain 3rd degree and 5th degree Bernstein polynomials for the function $f(x) = x^2$ on [0, 1]. **6**

8. a) State and prove Banach contraction mapping principle. **6**

b) Give an example of a mapping of [0, 1] onto itself that is not continuous and has no fixed point. **4**

c) Does there exists a function defined on (0, 1) that is continuous at each rational point of (0, 1) and discontinuous at each irrational point of (0, 1) ? Justify your answer. **6**

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M.A./M.Sc. (Semester – I) Examination, 2011
MATHEMATICS
MT-502 : Advanced Calculus
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Let $\vec{f}, \vec{g} : S \to \mathbb{R}^m$ where $S \subset \mathbb{R}^n$ be vector functions and $\vec{a} \in \mathbb{R}'^n$.

   Let $\lim_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{6}$ and $\lim_{\vec{x}\to\vec{a}} \vec{g}(\vec{x}) = \vec{c}$, then prove that $\lim_{\vec{x}\to\vec{a}} [f(\vec{x}).\vec{g}(\vec{x})] = \vec{b}.\vec{c}$. **6**

   b) Let $f(x, y) = \dfrac{x^2y^2}{x^2y^2 + (x-y)^2}$, where $x^2y^2+(x-y)^2 \neq 0$. Show that iterated limits of f(x,y) exist but f(x,y) does not tend to a limit as $(x, y) \to (0, 0)$. **5**

   c) Evaluate the directional derivative of $f(x, y, z) = \left(\dfrac{x}{y}\right)^z$ at (1,1,1) in the direction of $2\vec{i} + \vec{j} - \vec{k}$. **5**

2. a) Let $f: S \to \mathbb{R}$, $S \subset \mathbb{R}^n$ be a scalar field. Assume that the partial derivatives D,f,..., Dnf exist in some n-ball $B(\vec{a})$ and are bounded, then prove that f is continuous at $\vec{x} = \vec{a}$. **6**

   b) If a vector field $\vec{f}$ is differentiable at $\vec{a}$, then prove that $\vec{f}$ is continuous at $\vec{a}$. **5**

   c) Let Z be a function of u and v, where u = x^2-y^2-2xy and $v = y$. Find $(x+y)\dfrac{\partial z}{\partial x} + (x-y)\dfrac{\partial z}{\partial y}$. **5**

3\. a) State only the implicit function theorem. **2**

b) State only the chain rule for derivatives of vector fields in matrix form and explain the terms involved. **6**

c) Find the gradient vector for the scalar field f(x,y) = $x^2 + y^2\sin(xy)$ at a point $\left(1\pi/2\right)$. **3**

d) Let $\vec{g}(u,v,w) = uv^2w^2\vec{i} + w^2 zinv\vec{j} + u^2e^v\vec{k}$. Find the Jacobian matrix $D\vec{g}(1,2,3)$. **5**

4\. a) Define line integral of a vector field along the curve. Illustrate by an example that the value of the line integral is independent of the parametric representation of the curve. **6**

b) Evaluate $\int_c (x^2 - 2xy)dx + (y^2 - 2xy)dy$, where C is a path from (-2,4) to (1,1) along the parabola y = x^2. **4**

c) A particle of mass m moves along a curve under the action of the force field $\vec{f}$. If the speed of the particle at time t is v(t), its kinetic energy is defined to be $\frac{1}{2}mv^2(t)$. Prove that the change in kinetic energy in any time interval is equal to the work done by $\vec{f}$ during this time interval. **6**

5\. a) Let $\vec{f}$ be a vector field that is continuous on an open connected set S in $\mathbb{R}^n$, and assume that the line integral of $\vec{f}$ is independent of the path in S. Prove that $\vec{f}$ is a gradient of some scalar field. **8**

b) Is the function $\vec{f}(x,y) = x\vec{i} + xy\vec{j}$ gradient of some scalar field ? Compute $\int_C \vec{f}.d\vec{\alpha}$,where $C : \vec{\alpha}(t) = a\cos t\,\vec{i} + a\sin t\,\vec{j}, o \le t \le 2\pi$. **5**

c) Evaluate $\iint_Q xy(x+y)dxdy$, where $Q[0,1]\times[0,1]$. **3**

6. a) Prove that a continuous function on a rectangle is integrable. **8**

b) Let f be defined on a rectangle $Q[0,1]\times[0,1]$ as follows :

$$f(x,y)=\begin{cases}1-x-y \text{ if } & x+y\le 1\\ 0 & \text{otherwise}\end{cases}$$ **6**

make a sketch of the ordinate set of f over Q and compute volume of this ordinate set by double integral.

c) Transform the double integral $\int_0^1\left[\int_0^1 f(x,y)\,dy\right]dx$ to one or more iterated integrals in polar coordinates. **2**

7. a) Define a surface integral and explain the terms involved in it. **6**

b) Let S be a parametric surface whose vector representation is

$\vec{r}(u,v)=(u+v)\vec{i}+(u-v)\vec{j}+(1-2u)\vec{k}$.

Find the fundamental vector product and the unit normal vector to the surface. **5**

c) Transform the surface integral $\iint_S \left(\mathrm{curl}\vec{F}\right).\vec{n}\,ds$ to a line integral by the use of stoke's theorem, where $\vec{F}(x,y,z)=y^2\vec{i}+xy\vec{j}+xz\vec{k}$, where S is the hemisphere $x^2+y^2+z^2=1$, $z\ge 0$. **5**

8. a) State and prove the Divergence Theorem. **6**

b) Compute volume of the sphere of radius a by using triple integral . **5**

c) Use Green's theorem to evaluate the line integral $\oint_C y^2dx+xdy$ when C is square with vertices $(\pm 1,\pm 1)$. **5**

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M.A./M.Sc. (Semester – I) Examination, 2011
MATHEMATICS
MT – 504 : Number Theory
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. : 1) Answer **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) What is the highest power of 2 dividing 533! ? Also find the highest power of 3 dividing 533!. **6**

b) Explain Pollard ρ method for factorisation. **5**

c) Prove that the product of two primitive polynomials is primitive. **5**

2. a) What are the last two digits of 3^{541} ? **6**

b) State and prove the Chinese remainders theorem. **5**

c) Prove that $x^2 \equiv -1 \pmod{p}$ has a solution if and only if p = 2 or p is 4k + 1 type. **5**

3. a) If ϕ is Euler totient function on set of positive integers, then prove that it is multiplicative. **6**

b) Prove that 1 + i is a prime in $\mathbb{Z}[i]$. **5**

c) Characterize the set of positive integers n satisfying $\phi(2n) = \phi(n)$. **5**

4. a) If α and β are algebraic numbers, then prove that $\alpha + \beta$ and $\alpha\beta$ are algebraic numbers. **8**

b) Define a Transcendental Number. Prove that the number $\beta = \sum_{j=0}^{\infty} 10^{j!}$ is transcendental. **8**

P.T.O.

5. a) Determine whether the system $x \equiv 3 \pmod{10}$, $x \equiv 8 \pmod{15}$, $x \equiv 5 \pmod{84}$ has a solution and find them all, if any exist. **8**

b) State and prove the law of quadratic reciprocity. **8**

6. a) If ξ is an algebraic number of degree n, then every number in $\mathbb{Q}(\xi)$ can be written uniquely in the form $a_0 + a_1\xi + \ldots + a_{n-1}\xi^{n-1}$, where a_1's are rational numbers. **6**

b) For any odd prime p, let (a, p) = 1. Consider the integers a, 2a, 3a, ... {(p – 1)/2}a and their least positive residues modulo p. If n denotes the number of these residues that exceed p/2, then prove that $\left(\frac{a}{p}\right) = (-1)^n$. **5**

c) Find whether $x^2 - 4x + 31 \equiv 0 \pmod{41}$ has a solution. **5**

7. a) If $\left(\frac{p}{q}\right)$ is the Legendre symbol then evaluate (i) $\left(\frac{-23}{83}\right)$ (ii) $\left(\frac{51}{71}\right)$. **6**

b) Prove that the number of positive divisors of $n \in \mathbb{N}$ is odd if and only if n is perfect square. Also prove that Mobius mu function μ (n) is multiplicative. **5**

c) Find all integers x and y such that 10x – 7y = 17. **5**

8. a) Define unique factorization property. Does the field $\mathbb{Q}\left(\sqrt{-14}\right)$ have unique factorization property. Justify. **6**

b) Let a, b and m > 0 be given integers and put g = (a, m). Then prove that the congruence $ax \equiv b \pmod{m}$ has a solution if and only if g/b. Also prove that if this condition is met, then the solutions form an arithmetic progression with common difference m/g, giving g solutions (mod m). **5**

c) The reciprocal of a unit is a unit. The units of an algebraic number field form a multiplicative group. **5**

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M.A./M.Sc. (Semester – I) Examination, 2011
MATHEMATICS
M.T – 505 : Ordinary Differential Equations
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B. : i) Attempt **any five** questions.*
*ii) Figures to the **right** indicate **full** marks.*

1. a) If $y = y_1(x)$ is one solution of the differential equation $y'' + P(x)y' + Q(x)y = 0$ then find other solution. **6**

   b) Find the general solution of a differential equation $y'' - (m_1 + m_2)y' + m_1 m_2 y = 0$, where $m_1 \neq m_2$. **4**

   c) Show that e^x and e^{-x} are linearly independent solution of $y'' - y = 0$ on any interval. **4**

   d) Reduced the equation $\frac{d^2x}{dt^2} + P(t)\frac{dx}{dt} + Q(t)x = 0$ into an equivalent system of first order equation. **2**

2. a) Discuss the method of undetermine coefficients to find the solution of second order differential equation with constant coefficients. **8**

   b) Find the general solution of the differential equation $y'' - 2y = e^x \sin x$ by using method of variation of parameters. **6**

   c) Determine the nature of a point X = ∞ for the equation : $(1 - x^2)y'' - 2xy' + P(P+1)y = 0$. **2**

3. a) State and prove sturm separation theorem. **6**

   b) Show that zeros of functions a sin x + b cos x and c sin x + d cos x are distinct and occur alternately whenever ad – bc $\neq$ 0. **5**

   c) Find regular singular points of the differential equation : $x^2(x-2)^2 y'' + 2(x-2)y' + (x+3)y = 0$. **5**

P.T.O.

4. a) Find the general solution of a equation $\left(1-x^2\right)y'' - 2xy' + 2y = 0$ about $x = 0$ by power series method. **7**

b) Write 'tan x' in the form of a power series $\sum a_n x^n$ and verify this series is a solution of the equation $y' = y^2 + 1$. **5**

c) Find the indicial equation and its roots for the following differential equations :

i) $x^2 y'' + (\cos 2x - 1)\, y' + 2\, x\, y = 0$

ii) $2x^2 y'' + x\,(2x+1)\, y' - y = 0$. **4**

5. a) If m_1 and m_2 are roots of the auxiliary equation of the system

$$\frac{dx}{dt} = a_1 x + b_1 y$$

$$\frac{dy}{dt} = a_2 x + b_2 y$$

which are conjugate complex but not pure imaginary then prove that the critical point (0, 0) of the system is a spiral. **6**

b) Find critical points of

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - \left(x^3 + x^2 - 2x\right) = 0.$$ **5**

c) Show that (0, 0) is an asymptotically stable critical point of

$$\begin{cases} \dfrac{dx}{dt} = -y - x^3 \\[2ex] \dfrac{dy}{dt} = x - y^3 \end{cases}$$ **5**

6. a) Solve the system

$$\begin{cases} \dfrac{dx}{dt} = x - 2y \\ \dfrac{dy}{dt} = 4x + 5y \end{cases}$$ **8**

b) Prove that the function E (x, y) = $ax^2 + bxy + cy^2$ is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$. **4**

c) Find the equation of paths and sketch a few of the paths for a system

$\dfrac{dx}{dt} = e^y$; $\dfrac{dy}{dt} = e^y \cos x$. **4**

7. a) Discuss the method of solving a homogeneous linear system with constant coefficients :

$$\ddot{x}(t) = a_1 x + b_1 y$$

$$\ddot{y}(t) = a_2 x + b_2 y$$

when the auxiliary equation $m^2 - (a_1 + b_2)m + (a_1 b_2 - a_2 b_1) = 0$ has real roots. **8**

b) Perform three iterations of Picards method to find an approximate solution of the initial value problem :

$\dfrac{dy}{dx} = x^2 + y^2$; $y(0) = 0$. **6**

c) Find the general solution of a system

$\dfrac{dx}{dt} = y$; $\dfrac{dy}{dt} = x$. **2**

8\. a) Solve the following initial value problem by Picards method.

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x$$

given that x (0) = 1, y (0) = 0. **6**

b) Use the method of Frobenius to solve the differential equation $3xy'' + 2y' + y = 0$ about regular singular point 0 (zero). **8**

c) Show that F (x, y) = xy^2 does not satisfy a Lipschitz condition on any strip $a \le x \le b$ and $-\infty < y < \infty$. **2**

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M.A./M.Sc. (Semester – II) Examination, 2011
MATHEMATICS
MT – 602 : Differential Geometry
(2008 Pattern)

Time: 3 Hours | Max. Marks: 80

*N.B. : 1) Attempt **any five** questions.*
*2) Figures at **right** indicate **full** marks.*

1. a) Find the integral curve through p = (5, 6) for vector field defined by $X(x_1, x_2) = (x_2, x_1)$. **6**

   b) Define parametrized curve, velocity vector, divergence of smooth vector field with example. **6**

   c) Show that the gradient of f at $p \in f^{-1}(c)$ is orthogonal to all vectors tangent to $f^{-1}(c)$ at p. **4**

2. a) State and prove Lagranges multiplier theorem. **6**

   b) What is difference between tangent vector field and normal vector field on n-surface ? **4**

   c) Show that every connected n-surface in $\mathbb{R}^{n+1}$ has exactly two smooth unit normal vector field. **6**

3. a) Let S be compact, connected, oriented n-surface in $\mathbb{R}^{n+1}$ exhibited as level set $f^{-1}(c)$ of smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ with $\nabla f(p) \neq 0$ for all $P \in S$. Show that Guass map maps S onto unit sphere S^n. **12**

   b) Draw vector fields on $\mathbb{R}^2$ for $\overline{X}(p) = (p, x(p))$ where **4**

      i) X(p) = (1, 0) ii) X(p) = p.

4. a) True or false : Great circle is geodisic in 2-sphere justify. **5**

   b) Write a note on Levi Civita parrallelism. **5**

   c) Show that parrallel transport is one onto linear map. **6**

5. a) Find Weingarton map for n-sphere of radius r. **6**

b) True or false ? Justify. Weingarton map is self adjoint. **6**

c) Define : **4**

1) Length of parametrized curve

2) Differential 1-form.

6. a) Show that oriented plane curve is connected iff its global parametrization exists. **12**

b) Show that integral of an exact 1-form over a compact connected oriented plane curve is always zero. **4**

7. a) State and prove second derivative test for local minima and maxima. **12**

b) Define :

1) Tangent bundle

2) Guass kronecker curvature. **4**

8. a) State inverse function theorem. **4**

b) Prove that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point p such that the second fundamental form at p is definite. **12**

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M.A./M.Sc (Semester – II) Examination, 2011
MATHEMATICS
MT-603 : Groups and Rings (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : 1) Attempt ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let G be a finite group and H be a subgroup of G. Prove that order of H divides order of G. **5**

b) Suppose that a and b belong to a group G, a has odd order and $aba^{-1} = b^{-1}$. Show that $b^2 = e$. **5**

c) Let G be an abelian group with identity e and let n be some integer. Prove that the set of all elements of G that satisfy $x^n = e$ is a sub group of G. Give an example of a group G in which the set of all elements of G that satisfy $x^2 = e$ does not form a subgroup of G. **6**

2. a) Let H = {(1), (1, 2) (3, 4), (1,3) (2, 4), (1, 4) (2, 3)}. Show that H is a sub group of A_4. Further, show that H is a normal subgroup of A_4. Also, determine Aut (H). **10**

b) Let G and H be finite cyclic groups. Prove that $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. **6**

3. a) Let N be a normal subgroup of G and let H be a subgroup of G. If N is a subgroup of H, prove that H/N is a normal subgroup of G/N if and only if H is a normal subgroup of G. **6**

b) If K is a subgroup of G and N is a normal subgroup of G, prove that $K/(K \cap N)$ is isomorphic to $K\ N/N$. **6**

c) Let p, q be primes not necessarily distinct and G be a group having order pq. Show that $|Z(G)| = 1$ or pq. **4**

4. a) Let $\sigma = (1, 2, \ldots, n)$ and $\tau = (1, 2)$ be elements in S_n. Find the smallest subgroup of S_n which contains σ and τ. **8**

b) Let G be a finite group whose order is a power of a prime P. Then show that Z(G) has more than one element. Hence, or otherwise, show that every group of order P^2 is abelian. **8**

5. a) Find all Abelian groups upto isomorphism of order 360. **6**

b) Let G be a finite Abelian group of prime-power order and let a be an element of maximal order in G. Then prove that G can be written in the form $\langle a \rangle \times K$. **10**

6. a) Let H be a subgroup of a finite group G and $|H|$ be a power of a prime P. Prove that H is contained in same Sylow p-subgroup of G. **10**

b) Determine all the groups of order 99. **6**

7. a) Let $\mathbb{C}$ be the complex numbers and $M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

Prove that $\mathbb{C}$ and M are isomorphic under addition and that $\mathbb{C}^*$ and M*, the non-zero elements of M, are isomorphic under multiplication. **6**

b) Suppose s and t are relatively prime positive integers. Prove that U(st) is isomorphic to the external direct product of U(s) and U(t). **6**

c) Prove that every group of order 65 is cyclic. **4**

8. a) Let H be a normal subgroup of N and N be a normal subgroup of G. Then H is a normal subgroup of G. Is this statement true or false ? Justify. **6**

b) Show that $GL(2, \mathbb{Z}_2)$ is isomorphic to S_3. **6**

c) Determine all homomorphisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$. **4**

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B/II/11/315

M.A. / M.Sc. (Semester – II) Examination, 2011
MATHEMATICS
MT – 604 : Complex Analysis
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) For a given power series $\sum_{n=0}^{\infty} a_n z^n$ define the number R $0 \leq R \leq \infty$ by

$$\frac{1}{R} = \lim\sup |an|^{1/n}$$

Prove that :

i) If | z | < R, then the series converges absolutely.

ii) If 0 < r < R, then the series uniformly on {z : | z | $\leq$ r} **6**

b) Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function then prove that u has a harmonic conjugate. **5**

c) Find the fixed points of a dilation, a translation and the inversion on $\mathbb{C}\infty$ **5**

2. a) Define a branch of the Logarithm. Let f be a branch of Logarithm on an open connected subset G of $\mathbb{C}$. Show that the totality of branches of log z are functions. **6**

$$f(z) + 2Mik \quad k \in \mathbb{Z}$$

b) Define Mobius transformation. Prove that a Mobius transformation takes circle onto circle. **6**

c) Find the radius of convergance of $\sum_{n=0}^{\infty} \frac{n^3}{n!} z^n$. **4**

3. a) Let z_1, z_2, z_3, z_4 be four distinct points in $\mathbb{C}_\infty$. Show that the cross ratio (z_1, z_2, z_3, z_4) is a real number if and only if all four points lie on a circle. **6**

b) If $f : G \to \mathbb{C}$ is analytic and $\bar{B}(a;r) \subset G$ then prove that

$$f^{(n)}(a) = \frac{n!}{2Mi} \int_r \frac{f(w)}{(w-a)^{n+1}} dw \text{ where } r(t) = a+re^{it}\ 0 \le t \le 2M .$$ **6**

c) Let $r(t) = e^{it}$ for $0 \le t \le 2M$. Find $\int_r z^n dz_\delta$ for every integer n. **4**

4. a) Let $f:G \to \mathbb{C}$ be analytic and suppose $\bar{B}(a;r) \subset G$ $(r > 0)$. If $r(t) = a+re^{it}$, $0 \le t \le 2M$ then prove that $f(z) = \frac{1}{2Mi} \int_r \frac{f(w)}{w-z} dw$ for $|z-a| < r$. **8**

b) State and prove Fundamental Theorem of Algebra. **6**

c) State Maximum Modulus theorem. **2**

5. a) State and prove Morera's Theorem. **6**

b) If $r : [0, 1] \to \mathbb{C}$ is a closed rectifiable curve and $a \notin \{r\}$ then prove that

$\frac{1}{2Mi} \int_r \frac{dz}{z-a}$ is an integer. **5**

c) Evaluate the integral $\int_r \frac{e^{iz}}{z^2} dz$ where $r(t) = e^{it}\ 0 \le t \le 2M$. **5**

6. a) Let G be an open set and $f:G \to \mathbb{C}$ be a differentiable function then prove that f is analytic on G. **7**

b) Let G be a region and let f be an analytic function on G with zeros $a_1 \ldots, a_m$ (repeated according to multiplicity). If r is a closed rectifiable curve in G which does not pass through any point a_k and if $r \approx 0$ then prove that

$$\frac{1}{2Mi} \int_r \frac{f'(z)}{f(z)} dz = \sum_{k=1}^{M} \eta(r; a_k)$$ **5**

c) Let $\gamma(\theta) = \theta e^{i\theta}$ for $0 \le \theta \le 2M$ and $r(\theta) = 4M - \theta$ for $2M \le \theta \le 4M$.

Evaluate $\int_r \frac{dz}{z^2 + M^2}$. **4**

7\. a) State and prove Casorati Weierstrass Theorem. **6**

b) If f has a pole of order m at z = a and let $g(z) = (z - a)^m f(z)$ then prove that

$\text{Res}(f\,;a) = \frac{1}{(m-1)!}\, g^{(m-1)}(a)$. **5**

c) Show that for a > 1 $\int_0^M \frac{d\theta}{a + \cos\theta} = \frac{M}{\sqrt{a^2 - 1}}$. **5**

8\. a) State and prove Rouche's Theorem. **6**

b) If $|a| < 1$ then prove that φ_a is a one-one map of $D = \{z : |z| < 1\}$ onto itself ; the inverse of φ_a is φ_{-a}. Furthermore, φ_a maps ∂D onto ∂D, and $\varphi_a(a) = 0$,

$\varphi_a'(0) = 1 - |a|^2$, and $\varphi_a'(a) = \left(1 - |a|^2\right)^{-1}$. **5**

c) Let G be a bounded open set in $\mathbb{C}$ and suppose f is continuous function on $\overline{G}$ (closure of G) which is analytic in G. Then prove that
$\max\{|f(z)| : z \in \overline{G}\} = \max\{|f(z)| : z \in \partial G\}$. **5**

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M.A./M.Sc. (Semester – II) Examination, 2011
MATHEMATICS
MT – 606 : Object Oriented Programming Using C++
(2008 Pattern)

Time : 2 Hours Max. Marks : 50

N.B. : *1) Figures at* ***right*** *indicate* ***full*** *marks.*
2) Question ***one*** *is* ***compulsory****.*
3) Attempt ***any two*** *questions from Q.* ***2****, Q.* ***3*** *and Q.* ***4****.*

1. Attempt the following (**2** marks **each**) (**any 10**) : **20**

1) What are C++ keywords ? Give four example.

2) What is data encapsulation ?

3) Write output of following program.

```
    intx = 10 ;
    int & rx = x ;
cout < < “x = ” < < x < < endl ;
cout < < “rx = ” < < rx < < endl ;
```

4) Give example of union in C++.

5) Write syntax of output stream and input stream.

6) What is use of scope resolution operator ?

7) What are disadvantages of macros ?

8) Write the syntax of friend functions.

9) What is hierarchical inheritance ?

10) Write declaration syntax for overloading extraction operator.

11) Comment : A function cannot return by reference.

12) Explain the term “message passing”.

2. a) Write a program in C++ to find. Euclidean distance between two points in XY plane with output. **5**

   b) Compare dynamic memory management in C and C++. **5**

   c) Write a note on casting operators in C++. **5**

3. a) Write a note on general form of class declaration. **5**

   b) Write a note on function overloading. **5**

   c) Write a program in C++ to find simple interest. **5**

4. a) Write a note on Friend class. **5**

   b) What are rules to define constructor function ? **5**

   c) Write a note on local classes. **5**

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M.A./M.Sc. Semester – III Examination, 2011
MATHEMATICS
MT - 702 : Ring Theory (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : *i) Attempt* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Find all units in the ring $Z\left[\frac{1+\sqrt{-3}}{2}\right]$. Give an example of quadratic integer ring which has infinite number of units. **8**

b) If x is a nilpotent element of a commutative ring R then prove that,
   i) x is either zero or a zero divisor
   ii) 1 + x is unit
   iii) x + u is unit where u is unit in R. **8**

2. a) If R is an integral domain then prove that R[x] is an integral domain. What are the units in R[n] ? **8**

b) If R = z[x], ring of polynomials in x with integer coefficients and I is a collection of polynomials in R, whose terms are of degree at least 2 together with zero polynomial then show that I is an ideal and identify the quotient ring $\frac{R}{I}$.

Is $\frac{R}{I}$ an integral domain ? **8**

3. a) If A is a subring and B is an ideal of the ring R then prove that

$\frac{A+B}{B} \cong \frac{A}{A \cap B}$. Are the ring 2Z and 3Z isomorphic ? **8**

b) If R = z[x] ring of polynomials in x with integer coefficients, then show that
   i) $< x >$ is prime ideal but not maximal ideal.
   ii) $< z, x >$ is maximal ideal but not a principal ideal in z[x]. **8**

P.T.O.

4. a) If R is commutative ring and D is a non-empty subset of R such that $0 \notin D$ and does not contain any zero divisors and is closed under multiplicaiton then prove that there is a commutative ring Q with 1 such that Q contains R as a subring and every element of D is unit in Q. **8**

b) If A_1 and A_2 are coprime ideals of a commutative ring R with 1, then prove that

i) $A_1A_2 = A_1 \cap A_2$

ii) $\dfrac{R}{A_1 \cap A_2} \simeq \dfrac{R}{A_1} \times \dfrac{R}{A_2}$ **8**

5. a) Prove that every ideal in a Euclidean domain is principal. Is the converse true ? Justify. **8**

b) If R is a principal ideal domain and $I \neq 0$, is an ideal in R then prove that I is prime ideal iff I is a maximal ideal in R. **8**

6. a) If R is a principal ideal domain then prove that there exist a multiplicative Dede-Kind-Hasse norm on R. **8**

b) Determine all representation of integer 493 = 17.29 as a sum of two integer squares. **8**

7. a) If F is a field then prove that F[x] is a Euclidean domain. **8**

b) State Eisenstein criterian for irreducibility of polynomial. Prove that the polynomial $x^4 + 4x^3 + 6x^2 + 2x + 1$ is irreducible in z[x]. Find all monic irreducible polynomials of degree ≤ 3 in $F_2[x]$. **8**

8. a) If the ring R is Noetherian and has Krull dimension 0 then prove that R is Artinian. **8**

b) Prove that an Artinian integral domain is a field. **8**

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M.A./M.Sc. Semester – III Examination, 2011
MATHEMATICS
MT – 703 : Mechanics (Optional)
(New) (2008 Pattern)

Time : 3 Hours Max. Marks : 80

Instructions : *i) Attempt* ***any five*** *questions.*
ii) All questions carry ***equal*** *marks.*
iii) Figures to the ***right*** *indicate* ***maximum*** *marks.*

1. a) Explain the following terms :

i) Virtual work

ii) Degrees of freedom

iii) Holonomic constraints. **6**

b) Derive Lagranges equations of motion, from the D' Alembert's principle for conservative systems. **10**

2. a) Set up Lagrangian for a simple pendulum and write Lagrange's equations of motion. **5**

b) Show that the generalised momentum conjugate to a cyclic coordinate is conserved. **5**

c) Find degrees of freedom for

i) a rigid body

ii) conical pendulum

iii) simple pendulum. **6**

3. a) Let $L = \frac{m}{2}\left(\dot{q}^2 \sin^2 wt + q\dot{q}w \sin 2wt + q^2 w^2\right)$, is the Lagrangian of a particle.

Find the Hamiltonian. Is it conserved ? **6**

b) Prove the following transformation is canonical.

$$Q = \log\left(\frac{1}{q}\sin p\right), P = q\cot p.$$ **4**

c) Find a generating function for the canonical transformation :

$$Q = \log\left(1+\sqrt{q}\cos p\right), P = 2\left(1+\sqrt{q}\cos p\right)\sqrt{q}\sin p.$$ **6**

4. a) If the Hamiltonian H of the system is

$H\left(q_1, q_2, p_1, p_2\right) = q_1 p_1 - q_2 p_2 - aq_1^2 - bq_2^2$ then show that $q_1 q_2$ and

$\frac{p_1 - aq_1}{q_2}$ are constants of motion, where a, b are constants. **5**

b) Two masses m_1 and m_2, $m_1 > m_2$ are connected by an inextensible string of length 1 and the string passes over a weightless pulley. Show that the acceleration $a = \frac{m_1 - m_2}{m_1 + m_2} g$, where g is the gravitational acceleration. **5**

c) Write a note on brachistochrone problem. **6**

5. a) Find the stationary function of the integral $\int_0^4 \left(x\dot{y} - \dot{y}^2\right)dx$, where

$\dot{y} = \frac{dy}{dx}$, $y(0) = 0, y(4) = 3$. **4**

b) Show that Poisson brackets satisfy $[UV, W] = [U, W]V + U[V, W]$. **4**

c) State and prove the Jacobi identity in case of Poisson brackets. **8**

6. a) State and prove rotation formula. **6**

b) Explain Euler angles graphically. Find the matrix of transformation space set of axes to the body set of axes in terms of Euler angles. **10**

7. a) State and prove Euler's theorem on the motion of a rigid body. **6**

b) Show that infinitesimal rotations commute. **5**

c) Show that orthogonal transformations are length preserving. **5**

8. a) Define central force motion. Show that it is always planar. Further show that the areal velocity is constant. **4**

b) In case of three body motion, show that the center of motion of the 3 bodies either remains at rest or moves uniformly on a straight line. **4**

c) Derive the following differential equation for the path of the particle in the central force field

$$\frac{d^2u}{d\theta^2} + u = -\frac{f\left(\frac{1}{u}\right)}{mh^2u^2}$$ **8**

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M.A./M.Sc. (Semester – III) Examination, 2011

MATHEMATICS

MT–704 : Measure and Integration (New)

(2008 Pattern) (Optional)

Time: 3 Hours Max. Marks: 80

*N.B. : 1) Attempt **any five** questions.*

*2) Figures to the **right** indicate **full** marks.*

1\. a) Define following terms and give one example of each. **4**

i) σ-algebra

ii) Lebesgue measure

b) If E_i' s are measurable sets, then prove that $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu E_i$. **6**

c) Let $(X, \mathbb{B}, \mu)$ be a measure space and $Y \in \mathbb{B}$. Let $\mathbb{B}_Y$ consist of those sets of $\mathbb{B}$ that are conatined in Y and $\mu_y E = \mu E$ if $E \in \mathbb{B}_Y$. Then show that $(Y, \mathbb{B}_Y, \mu_y)$ is a measure space. **6**

2\. a) If c is a constant and the functions f and g are measurable then show that the functions f + c, cf, f + g, fg and $f \vee g$ are measurable. **6**

b) Let for each α in a dense set D of real numbers there is assigned a set $B_\alpha \in \mathbb{B}$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then prove that there is a unique measurable extended real valued function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \; B_\alpha$. **6**

c) Give an example of a function such that $|f|$ is measurable but f is not. **4**

3. a) State and prove Fatou’s Lemma. **6**

b) Define signed measure. If E be a measurable set such that $0 < \nu E < \infty$ then show that there is a positive set A contained in E with $\nu A > 0$. **6**

c) Show that if f is a non negative measurable function then f = 0 a.e. if and only if $\int f \, dx = 0$. **4**

4. a) Let $(X, \mathcal{B}, \mu)$ be a finite measure space and g an integrable function such that for some constant M,

$$\left|\int f\phi d\mu\right| \leq M\|\phi\|_q$$ **6**

for all simple functions ϕ. Then prove that $g \in L^q$.

b) Let F be a bounded linear functional on $L^p(\mu)$ with $1 < p < \infty$. Then show that there exist a unique element $g \in L^q$ such that **6**

$$F(f) = \int f \, g d\mu$$

and $\|f\| = \|g\|_q$.

c) Show that the set of numbers in [0, 1] which possess decimal expansions not containing the degit 5 has measure zero. **4**

5. a) i) Define an outer Measure μ^*

ii) Show that the class $\mathcal{B}$ of μ^*-measurable sets is a σ-algebra. **8**

iii) If $\bar{\mu}$ is μ^* restricted to $\mathcal{B}$, then prove that $\bar{\mu}$ is a complete measure on $\mathcal{B}$.

b) i) Define a measure on an algebra $\mathbb{G}$.

ii) If $A \in \mathbb{G}$ and $<A_i>$ is any sequence of sets in $\mathbb{G}$ such that $A \subset \cup_i = 1^\infty A_i$, then show that $\mu A \leq \sum_{i=1}^{\infty} \mu A_i$.

iii) If $A \in \mathbb{G}$, then prove that A is measurable with respect to μ^*. **8**

6. a) Let $(X, \mathbb{G}, \mu)$, and $(Y, \mathbb{B}, \nu)$ be two σ-finite measure spaces and let f be a non-negative measurable function on $X \times Y$. Then prove the following : **10**

i) For almost all x the function f_x defined by $f_x(y) = f(x, y)$ is a measurable function on Y.

ii) For almost all y the function f^y defined by $f^y(x) = f(x, y)$ is a measurable function on X.

iii) $\int_Y f(x, y) d\nu(y)$ is a measurable function on X.

iv) $\int_X f(x, y) d\mu(x)$ is a measurable function on Y.

v) $\int_x \left[\int_y f \, d\nu \right] d\mu = \int_{x \times y} f d(\mu \times \nu) = \int_Y \left[\int_x f \, d\mu \right] d\nu$

b) Define an inner measure μ_*. Let B be a μ^* - measurable set with $\mu^* B < \infty$ then show that $\mu_* B = \mu^* B$. **6**

7. a) Define a Caratheodory outer measure. If μ^* is a Caratheodory outer measure with respect to Γ (a set of real valued functions on X) then show that every function in Γ is μ^* measurable. **8**

b) Let (X, ρ) be a metric space and let μ^* be an outer measure on X with the property that $\mu^*(A \cup B) = \mu^* A + \mu^* B$ whenever ρ (A, B) > 0 then show that every closed set is measurable with respect to μ^*. **4**

c) Let $\{A_n\}$ be Borel sets and let α_n be the Hausdorff dimension of A_n. Find the Hausdorff dimension of $A = \bigcup_{n=1}^{\infty} A_n$. **4**

8. a) Let (X, $\mathbb{B}$, μ) be a σ-finite measure space and let ν be a measure defined on $\mathbb{B}$ which is absolutely continuous with respect to μ then prove that there exist a non negative measurable function f such that **8**

$$\nu E = \int_E f \, d\mu$$

for each set E in $\mathbb{B}$.

b) With usual notation prove that **8**

$\mu_* E + \mu_* F \le \mu_*(E \cup F) \le \mu_* E + \mu^* F \le \mu^*(E \cup F) \le \mu^* E + \mu^* F$ where E and F are disjoint sets.

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M.A./M.Sc. (Semester – III) Examination, 2011
MATHEMATICS (Optional)
MT–704 : Mathematical Methods – I (Old)
(2005 Pattern)

Time: 3 Hours Max. Marks: 80

*N.B. : 1) Attempt **any five** questions.*

*2) Figures to the **right** indicate **full** marks.*

1. a) Find whether the following series converges or diverges. **4**

$$\sum_{n=3}^{\infty} \frac{\sqrt{2n^2 - sn + 1}}{4n^3 - 7n^2 + 2}$$

b) Prove that an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ is convergent. **4**

c) Find the series expansion : **8**

i) e^{-x^2}

ii) $e^{\tan x}$

2. a) Explain comparison test , integral test, ratio test for convergence of series of positive terms. **8**

b) Find the interval of convergence of the Power series. **4**

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{3^n}$$

c) Evaluate $\log_e \sqrt{(1+x)/(1-x)}$ – tanx at x = 0.0015. **4**

3. a) Define even function and odd function, also sketch the graph and give geometrical interpretation of the functions, f(x) = x and g(x) = cosx. **8**

b) Find the Fourier series of the function . **8**

$$f(x)=\begin{cases} x, & 0<x<1 \\ -2, & 1<x<2 \end{cases}.$$

4. a) Define Gamma function and Beta function. **4**

b) Show that, $B(p,q)=\dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, with usual notation. **8**

c) Prove that, B(p, q) = B(q, p). **4**

5. a) Express the following integrals as Beta function and evaluate it.

i) $J=\int_0^1 \dfrac{dx}{\sqrt{1-x^3}}$

ii) $I=\int_0^2 \dfrac{x^2\,dx}{\sqrt{2-x}}$ **8**

b) Prove that, $\Gamma(p)\Gamma(1-p)=\dfrac{\pi}{\sin\pi p}$. **8**

6. a) Prove that, the Rodrigues formula.

$P_m(x)=\dfrac{1}{2^m m!}\dfrac{d^m}{dx^m}(x^2-1)^m$, with usual notation. **8**

b) i) Show that, erf (–x) = – erf (x)

ii) Show that, erf (∞) = 1. **8**

7. a) If L [f(t)] = F(s) and

$$F(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$ then show that $L[F(t)] = e^{-as} F(s)$. **4**

b) State and prove convolution theorem for Fourier transform. **8**

c) Evaluate, $\int_0^\infty te^{-3t} \sin t \, dt$ **4**

8. a) Solve differential equation by Laplace transform. **8**

$$y''(t) + 4y'(t) + 13y(t) = \frac{1}{3} e^{-2t} \sin 3t$$

$y(0) = 1$, $y'(0) = -2$.

b) Find the Fourier transform of **8**

i) $F(x) = 2x$, $0 \le x \le 4$

ii) $F(x) = nx - x^2$, $0 \le x \le n$

M.A./M.Sc. (Semester – III) Examination, 2011
MATHEMATICS (2005 Pattern) (Optional)
MT-706 : Numerical Analysis (Old Course)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Answer **any five** questions.*
*2) Figures to the **right** indicates **full** marks.*
*3) **Use** of unprogrammable, scientific calculator is **allowed**.*

1. A) Assume that g(x) and $g^1(x)$ are continuous on a balanced interval $(a,b) = (P-\delta, P+\delta)$ that contain the unique fixed point P and that starting value P_0 is chosen in the interval. Prove that if $\left|g^1(x)\right| \leq K < 1 \forall x \in [a,b]$ then the iteration $P_n = g(P_{n-1})$ converges to P and if $\left|g^1(x)\right| \succ 1 \forall x \in [a,b]$ then the iteration $P_n = g(P_{n-1})$ does not converges to P. **6**

B) Investigate the nature of iteration in part (a) when $g(x) = 3(x - 2.25)^{1/2}$

i) Show that P = 4.5 is the only fixed points.

ii) Use $P_0 = 4.4$ and compute p_1, p_2, p_3, p_4. **5**

C) Start with the interval [1,1.8] and use the Bisection method to find an interval of width h = 0.05 that contains a solution of the equation $\exp(x) - 2 - x = 0$. **5**

2. A) Assume that $f \in c^2[a,b]$ and exist number $p \in [a,b]$ where f(p) = 0. If $f'(p)$ # 0 prove that there exist a $\delta \succ 0$ such that the sequence $\{p_k\}$ defined by iteration $p_k = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})}$ for k = 1, 2... converges to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$. **6**

B) Let $f(x) = x^2 - 2x - 1$

i) Find Newton-Raphson formula.$g(p_{k-1})$.

ii) Start with $p_0 = 2.5$ and compute p_1, p_2, p_3.

iii) Start with $p_0 = -0.5$ and compute p_1, p_2, p_3. **5**

C) Solve the system of equation

$$\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

Using the Gauss Elimination Method with partial Pivoting. **5**

3. A) Explain Gaussian Elimination method for solving a system of m equations in n knows. **6**

B) Find the Jacobin J(X, Y, Z) of order 3 × 3 at the point (1,3,2) for the functions

$f_1(X,Y,Z)=X^3-Y^2+Y-Z^4$, $f_2(X,Y,Z)=XY+YZ+XZ$, $f_3(X,Y,Z)=\frac{Y}{XZ}$. **5**

C) Consider $P(x) = -0.02x^3 + 0.1x^2 - 0.2x + 1.66$ which passes through the four points (1,1.54) (2,1.5) (3,1.42) and (5,0.66).

Find :

a) P(4)

b) P'(4)

c) The integral P(x) taken over [1,4]. **5**

4. A) Assume that $f \in C^{N+1}[a,b]$ and x_0, $x_1 \ldots x_N \in [a,b]$ are N + 1 nodes. If $x \in [a,b]$ them prove that $f(x) = P_N(x) + E_N(x)$

where $P_N(x)$ is a polynomial that can be used to approximate f(x) and $E_N(x)$ is the corresponding error in the approximation. **6**

B) Consider the system :

$5x - y + z = 10$

$2x + 8y - z = 11$

$-x + y + 4z = 3$ $\quad\quad P_0 = 0$

And use Gauss-Seidel iteration to find P_1, P_2, P_3. Will this iteration convergence to the solution. **5**

C) Find the triangular factorization A = LU for the matrix

$$A = \begin{bmatrix} -5 & 2 & -1 \\ 1 & 0 & 3 \\ 3 & 1 & 6 \end{bmatrix}.$$ **5**

5. A) Assume that $f \in C^5[a,b]$ and that $x-2h, x-h, x, x+h, x+2h \in [a,b]$

prove that $f'(x) \approx \dfrac{-f(x)+2h+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$. **6**

B) Let $f(x) = x^3$ find approximation for $f'(2)$. Use formula in Part (a) with $h = 0.05$. **5**

C) Use Newton's method with the starting value $(p_0,q_0) = (2.00, .25)$ compute (p_1,q_1) (p_1,q_1) (p_2,q_2) for the nonlinear system;

$x^2 - 2x - y + 0.5 = 0$, $x^2 + 4y^2 - 4 = 0$. **5**

6. A) Assume that $X_j = X_0 + h_j$ are equally spaced nodes and $f_j = f(x_j)$. Derive the Quadrature formula $\int_{x_0}^{x_2} f(x) \approx \dfrac{h}{3}(f_0 + 4f_1 + f_2)$. **6**

B) Let $f(x) = \dfrac{8x}{2^x}$

Use cubic Lang ranges interpolation based on nodes $x_0 = 0, x_1 = 1 x_2 = 2$, $x_3 = 3$ to approximate $f(1.5)$. **5**

C) Consider $f(x) = 2 + \sin(2\sqrt{x})$. Investigate the error when the composite trapezoidal rule is used over [1, 6] and the number of subinterval is 10. **5**

7. A) Use Euler's method to solve the I V P

$y' = -ty$ over [0, 0.2] with $y(0) = 1$. Compute y_1, y_2 with $h = 0.1$

Compare the exact solution $y(0.2)$ with approximation. **8**

B) Use the Runge-kutta method of order $N = 4$ to solve the I.V.P. $y' = t^2 - y$ over [0,0.2] with $y(0) = 1$, (take $h = 0.1$)

Compare with $y(t) = -e^{-t} + t^2 - 2t + 2$. **8**

8. A) Use power method to find the dominant Eigen value and eigen vector for the Matrix $A = \begin{bmatrix} 0 & 11 & -5 \\ -2 & 17 & -7 \\ -4 & 26 & -10 \end{bmatrix}$. **8**

B) Use Householder's method to reduce the following symmetric matrix to trigonal form

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. **8**

M.A./M.Sc. (Sem. – IV) Examination, 2011
MATHEMATICS
MT-801 : Field Theory (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : *i) Attempt* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let $F \subseteq E \subseteq K$ be fields. If [K:E] and [E:F] are finite, then prove that [K:F] is finite and [K:F] = [K:E] [E:F]. **8**

   b) If $f(x) = x^2 - x - | \in \mathbb{Z}_3[x]$, then show that f(x) is irreducible over $\mathbb{Z}_3$. **3**

   c) Show that there exists an extension K of $\mathbb{Z}_7$ with 49 elements. **5**

2. a) If $E = F(u_1, u_2, ---, u_k)$ is finitely generated field over F such that each of $u_1, u_2, ---, u_k$ is algebraic over F. Then prove that E is finite extension over F. Is E algebraic over F ? Justify. **6**

   b) Let E be an algebraic extension of F, and let $\sigma : E \to E$ be an embedding of E into itself over F. Then prove that σ is onto and hence an automorphism of E. **4**

   c) Find a suitable number α such that $\mathbb{Q}\left(\sqrt{3}, i\right) = \mathbb{Q}(\alpha)$. **6**

3. a) Let F be a field, and let $\sigma : F \to L$ be an embedding of F into an algebraically closed field L. Let $E = F(\alpha)$ be an algebraic extension of F. Then prove that σ can be extended to an embedding $\eta : E \to L,$ and the number of such extensions is equal to the number of distinct roots of the minimal polynomial of α. **8**

   b) Show that $x^3 - 2 \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$. Find (if it exists) an extension K of $\mathbb{Q}$ having all roots of $x^3 - 2$. Also find $[K:\mathbb{Q}]$. **8**

4. a) Prove that any irreducible polynomial f(x) over a field of characteristic zero has simple roots. Next prove that any irreducible polynomial f(x) over a field F of characteristic $p \neq 0$ has multiple roots if and only if there exists $g(x) \in F[x]$ such that $f(x) = g(x^p)$. **8**

b) Construct splitting field over $\mathbb{Q}$ of $x^6 - 1$, also find the degree of the extension. **3**

c) Which of the following are extensions are normal over $\mathbb{Q}$?

i) $\mathbb{Q}\left(5\sqrt{7}\right)$

ii) $\mathbb{Q}\left(\sqrt{-1}\right)$ and

iii) $\mathbb{Q}(x)$,

Where x is not algebraic over $\mathbb{Q}$. **5**

5. a) Prove that the prime field of a field F is either isomorphic to $\mathbb{Q}$ or to $\mathbb{Z}p$, p-prime. **6**

b) Prove that the multiplicative group of non-zero elements of a finite field is cyclic. **6**

c) Find irreducible polynomial of degree 3 over $\mathbb{Z}_2$. **4**

6. a) Let E be a finite extension of a field F. Then prove that $E = F(\alpha)$ for some $\alpha \in E$ if and only if there are only finite number of intermediate fields between F and E. **8**

b) Find $\alpha \neq \sqrt{2} + \sqrt{3}$ such that $\mathbb{Q}\left(\sqrt{2}, \sqrt{3}\right) = \mathbb{Q}(\alpha)$. **4**

c) If E is finite extension of a field F, and E is splitting field of a polynomial in F[x], then E is normal extension of F, prove. **4**

7. a) Let E be a finite separable extension of a field F, and let $H < G(E/F)$. Then prove that $G\left(E/E_H\right) = H$ and $[E:E_H] = /G\left(E/E_H\right)$. **6**

b) Let $E = \mathbb{Q}\left(3\sqrt{2}\right)$, prove that $G\left(E/\mathbb{Q}\right)$ is trivial. **6**

c) Let F be a field of characteristic $\neq 2$. Let $x^2 - a \in F[x]$ be an irreducible polynomial over F. Then show that Galois group of $x^2 - a$ is of order 2. **4**

8. a) A polynomial $f(x) \in F[x]$ is solvable by radicals over F if its splitting field E over F has solvable Galois group $G\left(E/F\right)$, prove this statement. **8**

b) Prove that the group of automorphisms of a field F with p^n elements is cyclic of order n. Find its generator. **8**

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M.A./M.Sc. (Semester – IV) Examination, 2011
MATHEMATICS
MT – 802 : Combinatorics (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B.: *i) Attempt* ***any five*** *questions.*
ii) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) How many ways are there to pick 2 different cards from a standard 52 card deck such that :
   i) The first card is an ace and the second card is not a Queen ?
   ii) The first card is a spade and the second card is not a Queen ? **6**

   b) There are 10 people at a party. How many different ways are there to pair them off into a collection of 5 pairings ? **5**

   c) How many ways are there to arrange the seven letters in the word SYSTEMS ? In how many of these arrangements do the '3 S' appear consecutively ? **5**

2. a) If a coin is flipped 10 times, what is the probability of 8 or more heads ? **6**

   b) How many ways are there to fill a box with a dozen doughnuts chosen from five different varieties with the requirement that atleast one doughnut of each variety is picked ? **5**

   c) How many integer solutions are there to $x_1 + x_2 + x_3 = 0$ with $x_i \geq -5, i = 1, 2, 3$? **5**

3. a) Find the coefficient of x^{16} in $(x^2 + x^3 + x^4 + \ldots)^5$. What is the coefficient of x^r ? **8**

   b) How many ways are there to select 25 toys from seven types of toys with between two and six of each type ? **8**

4. a) Find the generating function of a_r, the number of ways to select r balls from a pile of three green, three white, three blue and three gold balls. **6**

   b) Prove using a combinatorial argument $\binom{2n}{2} = 2\binom{n}{2} + n^2$. **5**

   c) Find a generating function for a_r, the number of ways that we can choose 2 cents, 3 cents and 5 cents stamps adding to a net value of r cents. **5**

5. a) Find a recurrence relation for the number of ways, to arrange n dominos to fill a 2 – by – n checkerboard. **5**

   b) Find a recurrence relation for the number of sequences of 1's, 3's and 5's whose terms sum to n. **6**

   c) Solve the recurrence relation $a_n = 2a_{n-1} + 1$ with $a_1=1$. **5**

6. a) Use generating functions to solve the recurrence relation $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 1$, $a_2 = 2$. **6**

   b) Show that if n+1 distinct numbers are chosen from 1, 2, ..., 2n, then two of the numbers must always be consecutive integers. **4**

   c) Suppose the numbers 1 through 10 are randomly placed around a circle. Show that the sum of some set of three consecutive numbers must be atleast 17. **6**

7. a) State and prove the Inclusion Exclusion Principle. **8**

   b) What is the probability that if n people randomly reach into a dark closet to retrieve their hats, no person will pick his own hat ? **8**

8. a) How many ways are there to send 6 different birthday cards, denoted by C_1, C_2, C_3, C_4, C_5, C_6 to three aunts and three uncles, denoted by A_1, A_2, A_3, U_1, U_2, U_3, if aunt A_1 would not like the cards C_2 and C_4; if A_2 would not like the cards C_1 or C_5; if A_3 likes all cards; if U_1 would not like the cards C_1 and C_5; if U_2 would not like C_4; and if U_3 would not like C_6 ? **6**

   b) How many positive integers ≤ 70 are relatively prime to 70 ? **5**

   c) How many n digit decimal sequences are there in which the digits 1, 2, 3 all appear ? **5**

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M.A./M.Sc. (Semester – IV) Examination, 2011
MATHEMATICS
MT – 802 : Hydrodynamics (Old Course)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Attempt **any five** questions.*
*2) **Each** question carry **equal** marks.*

1. a) Derive Bernoulli's equation. **8**

b) Derive equation of mass conservation in Eulerian description. **6**

c) Write a note on continuum hypothesis. **2**

2. a) A two dimensional incompressible flow field has the x-component of velocity given by $u = \frac{x}{(x^2 + y^2)^{3/2}}$. Determine y-component of velocity. Is this flow irrotational ? Justify your answer. **8**

b) For steady irrotational flow in two dimensions derive the relation between velocity potential and stream function in polar co-ordinate system. **8**

3. a) If vorticity is zero, circulation is zero. Is the converse true ? Justify your answer. **10**

b) A three dimensional velocity field is given by $u = xy^2$, $v = \frac{y^3}{3}$, $w = \frac{1}{2}xyz^2$. Determine acceleration. **6**

4. a) State and prove circle theorem and hence obtain the image of a source in a circle. **10**

b) Given the stream function $\psi = \frac{y^3}{3} - x^2 y$

i) What is the velocity vector that describes the flow ?

ii) Find velocity potential. **6**

5. a) State and prove Blasius theorem. **10**

b) Discuss the flow for which complex potential is $W = Uz^n$ where U is constant and $n > 0$, a real number. **6**

6. a) Two point vortices each of strength K are situated at $(\pm a, 0)$ and a point vortex is of strength $-\frac{k}{2}$ is situated at the origin. Show that the liquid motion is stationary. Determine stagnation points. **8**

b) Obtain the relation between stress and rate of strain components. **8**

7. a) When a cylinder of any shape is placed in a uniform stream of speed U, show that the resultant thrust on the cylinder is a lift of magnitude $K\rho U$ per unit length and is at right angles to the stream where K is circulation around a cylinder. **10**

b) Show that stress tensor is symmetric. **6**

8. a) Write a note on Karman vortex street. **6**

b) Three parallel rectilinear vortices of the same strength K and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices all move around the same cylinder with uniform speed in time $\frac{2\pi a^2}{3K}$. **10**

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M.A./M.Sc. (Semester – IV) Examination, 2011
MATHEMATICS
MT-803 : Differential Manifolds (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

*N.B.: 1) Attempt **any five** questions.*
*2) Figures to the **right** indicate **full** marks.*

1. a) Let W be a linear subspace of $\mathbb{R}^n$ of dimension K. Prove that there is an orthogonal transformation $h : \mathbb{R}^n \to \mathbb{R}^n$ that carries W onto the subspace $\mathbb{R}^K \times O$ of $\mathbb{R}^n$. **8**

b) If $X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ a & b \end{pmatrix}$, then find V(X). **4**

c) Give an example of a 1-manifold in $\mathbb{R}^3$. **4**

2. a) Justify whether **true** or **false** :

If U is open in H^K and if $p = \alpha(x_0)$ for $x_0 \in H_+^K$, then p is an interior point of M. **6**

b) If $f(X, Y) = x_2y_3 - y_2x_3$, then is f an alternating tensor on $\mathbb{R}^4$? **5**

c) Find the length of the parametrized curve $\alpha(t) = (a \cos t, a \sin t)$ for $0 < t < 3\pi$. **5**

3. a) Let U be an open set in $\mathbb{R}^n$ and $f : V \to \mathbb{R}$ be of class C^r. Let $N = \{x \in \mathbb{R}^n : f(x) \geq 0\}$. Show that N is an n-manifold in $\mathbb{R}^n$. **6**

b) Let X,Y,Z $\in \mathbb{R}^5$. If $F(X,Y,Z) = 2x_2y_2z_1 + x_1y_5z_4$, then write AF in term of elementary alternating tensors. **6**

c) Show that the upper hemisphere of $S^{n-1}(a)$, defined by the equation $E_+^{n-1} = S^{n-1}(a) \cap H^n$ is an (n – 1) manifold. **4**

P.T.O.

4. a) State and prove Green's theorem. **8**

b) Define differential of a K-form and prove that d(dw) = 0. **8**

5. a) If α is an l-form in $\mathbb{R}^n$, prove that $\alpha^*(dw) = d(\alpha^* w)$. **6**

b) Let A = $\mathbb{R}^2$– 0; consider the 1-form in A defined by

$$w = \frac{x\,dx + y\,dy}{x^2 + y^2}$$

Show that w is closed. **6**

c) If $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ is given by $\alpha(u, v) = (u, u + v, v^2)$ and $w = xy\,dx + zz\,dy - y\,dz$, then compute $\alpha^*(dw)$. **4**

6. a) Define an oriented K-manifold. For K > 1, if M is an orientable K-manifold with non-empty boundary, then prove that ∂M is also orientable. **6**

b) Let $M = \{X \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$.

Find the tangent space $T_p(M)$ at $p = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$. **5**

c) Consider w = xy dx + 3 dy – yz dz and $\eta = x dx - yz^2 dy + 2x\,dz$. Verify that $d(w \wedge \eta) = (dw) \wedge \eta - w \wedge d\eta$. **5**

7. a) Define the term 'induced orientation'. **4**

b) State generalized Stokes's theorem. **4**

c) Let A = $(0, 1)^2$. Let $\alpha : A \to IR^3$ be given by $\alpha(u,v) = (u,v, u^2 + v^2 + 1)$. Let Y be the image set of α. Evaluate

$$\int_{Y_\alpha} x_2 dx_2 \wedge dx_3 + x_1 x_3 \, dx_1 \wedge dx_3 .$$ **8**

8. a) Let A be an open set in IR^K and $\alpha : A \to IR^3$ be of class C^∞. If w is an l-form defined in an open set of IR^n containing $\alpha(A)$, then prove that $\alpha^*(dw) = d(\alpha^* w)$. **8**

b) Show that the n-ball $B^n(a)$ is an n-manifold in IR^n of class C^∞. **4**

c) Justify whether **true** or **false** :

The 2-sphere S^2 is a 2-manifold in IR^3. **4**

M.A./M.Sc. (Sem. – IV) Examination, 2011
MATHEMATICS
MT-805 : Lattice Theory (New)
(2008 Pattern)

Time : 3 Hours Max. Marks : 80

N.B. : *1) Attempt* ***any five*** *questions.*
2) Figures to the ***right*** *indicate* ***full*** *marks.*

1. a) Let A be the set of all subgroups of a group G; for X, Y $\in$ A, set X $\leq$ Y to mean X $\subseteq$ Y. Prove that $\langle A;\leq\rangle$ is a lattice. **5**

   b) Let L be a lattice. Then prove that I is a proper ideal of L if and only if there is a onto join-homomorphism $\phi : L \to C_2$ such that $I = \phi^{-1}(0)$. **5**

   c) Define a congruence relation on a lattice L and if θ is a congruence relation of L then prove that for every $a \in L, [a]\theta$ is a convex sublattice. **6**

2. a) If a lattice L is finite, then prove that $L \simeq Id(L)$, the ideal lattice of L. **4**

   b) Let L be a lattice. Then prove that a homomorphic image of L is isomorphic to a suitable quotient lattice of L. **7**

   c) Show that an ideal P is a prime ideal of a lattice L if and only if L\P is a dual ideal. **5**

3. a) Prove that every distributive lattice is modular, but not conversely. Find the smallest modular but not distributive lattice. **4**

   b) Let L be a distributive lattice with 0. Show that Id(L), the ideal lattice of L, is pseudocomplemented. **4**

   c) Prove that a lattice is modular if and only if it does not contain a pentagon (N_5) as a sublattice. **8**

4. a) Prove that a lattice L is a chain if and only if its every ideal is a prime ideal. **6**

b) Show that the following inequalities hold in any lattice. **4**

1) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$;

2) $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee (x \wedge z))$.

c) Prove that a lattice L is distributive if and only if for any two ideals I, J of L, $I \vee J = \{i \vee j \mid i \in I, j \in J\}$. **6**

5. a) Let L be a pseudocomplemented lattice. Show that $a^{**} \vee b^{**} = (a \vee b)^{**}$. **5**

b) Prove that every meet homomorphism is isotone. Is the converse true ? Justify. **4**

c) State and prove Nachbin theorem. **7**

6. a) Let L be a finite distributive lattice. Then show that the map $\phi : a \mapsto r(a)$ is an isomorphism between L and H(J(L)), the hereditary subsets of the set of join-irreducibles of L. **8**

b) Let L be a distributive lattice, let I be an ideal, let D be a dual ideal of L, and let $I \cap D = \phi$. Then prove that there exists a prime ideal P of L such that $P \supseteq I$ and $P \cap D = \phi$. **8**

7. a) State and prove Jordan-Hölder Theorem for semimodular lattices. **8**

b) Prove that every modular lattice is semimodular but not conversely. **5**

c) Prove that every complete lattice is bounded. Is the converse true ? Justify your answer. **3**

8. a) State and prove Fixed-Point Theorem for complete lattices. **6**

b) Prove that any lattice can be embedded into its ideal lattice. **4**

c) Let L be a lattice, let P be a prime ideal of L, and let a, b, c ∈ L. Prove that if $a \vee (b \wedge c) \in P$ then $(a \vee b) \wedge (a \vee c) \in P$. **6**
